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004

POSITIVE DISKS WITH PRESCRIBED MEAN CURVATURE ON THE BOUNDARY*

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1. Introduction. By a positive disk we mean a positive curvature metric $g = g_{ij}du^i du^j$, defined on the closed unit disk $\bar{D} = \{(u^1)^2 + (u^2)^2 \leq 1\}$. In the sequel we always denote it by $(\bar{D}, g)$. Recently, there have been some papers devoted to the boundary value problems for isometric embedding of positive disks into R^3 with prescribed vertical component of position vector, for example, [PO] [DE] [HO2] and [HO3]. The present paper is devoted to the Neumann problem for isometric embedding of positive disks which is posed by Yau in [Y4]. Given a smooth positive disk $(\bar{D}, g)$ and a positive function $h \in C^\infty(\partial D)$, can we find a smooth surface

$$\vec{r} : \bar{D} \mapsto R^3 \text{ such that } d\vec{r}^2 = g \tag{1.1}$$

$$\text{with the prescribed mean curvature } h \text{ on } \vec{r}(\partial D)$$

Before describing our results, we first introduce an invariant related to umbilical points of surfaces in R^3. Suppose that the given metric is of the form

$$g = Edx^2 + 2Fdxdy + Gdy^2,\ (x, y) \in \bar{D}. \tag{1.2}$$

Let $\vec{r}$ be a smooth isometric immersion of $(\bar{D}, g)$ with the second fundamental form

$$II = Ldx^2 + 2Mdxdy + Ndy^2 \text{ for } (x, y) \in \bar{D} \tag{1.3}$$

DEFINITION. *If $\vec{r}$ is of no umbilical points on ∂D, with*

$$\sigma = 2(EM - FL) + \sqrt{-1}(GL - EN)$$

the winding number of σ on ∂D is called the index of the umbilical points of the surface $\vec{r}$ and denoted by $Index(\vec{r})$ or $Index(\sigma)$.

Obviously, this definition makes sense since p is an umbilical point if and only if $\sigma(p) = 0$. We shall show that the definition of the index of the umbilical points is coordinate-free and hence, an invariant of describing umbilical points of surfaces.

The problem about the realization of a positive disk into R^3 seems to have some obstructions. For details, refer to Gromov's counter example [GR] which is an analytic positive disk not admitting any C^2 isometric immersion. On the other hand, to the author's knowledge the unique known sufficient condition for a smooth positive disk to be smoothly embedded into R^3 is the geodesic curvature of the boundary positive. Therefore the following hypothesis for the solvability is natural. Assume that

$$(\bar{D}, g) \text{ admits a smooth isometric immersion } \vec{r}_0 \text{ in } R^3 \tag{1.4}$$

The main result of the present paper is as follows.

THEOREM A. *If $(\bar{D}, g)$ is a smooth positive disk satisfying (1.4), then for any nonnegative integer n and arbitrary $(n+1)$ distinct points $p_0 \in \partial D$, $p_1, \ldots, p_n \in D$,*

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