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BIEBERBACH THEOREMS FOR SOLVABLE LIE GROUPS*

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1. Introduction. Let G be a Lie group and let $\mathrm{Aut}(G)$ be the group of continuous automorphisms of G. The group $\mathrm{Aff}(G)$ is the semi-direct product $\mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G)$ with multiplication

$$(a, \alpha) \cdot (b, \beta) = (a \cdot \alpha(b), \alpha\beta).$$

It has a Lie group structure and acts on G by

$$(a, \alpha) \cdot x = a \cdot \alpha(x)$$

for all $x \in G$. With the linear connection on G defined by the left invariant vector fields, it is known that $\mathrm{Aff}(G)$ is the group of connection-preserving diffeomorphisms of G, see [KT] Proposition 2.1.

TERMINOLOGY. *In this paper we will use the term lattice of a Lie group G, to denote a discrete cocompact subgroup of G. Further, we will say that an automorphism α of a Lie group G is unipotent if and only if its differential $d\alpha$ (in the automorphism group of the corresponding Lie algebra $\mathfrak{G}$) is unipotent. Analogously we will speak of an element acting unipotently on a Lie group G.*

For $G = \mathbb{R}^n$, the following three theorems have been proven by Bieberbach. See [W] or [C]:

THEOREM 1′. *Let $\pi \subset \mathbb{R}^n \rtimes \mathrm{O}(n)$ be a lattice. Then $\Gamma = \pi \cap \mathbb{R}^n$ is a lattice of $\mathbb{R}^n$, and Γ has finite index in π.*

THEOREM 2′. *Let $\pi, \pi' \subset \mathbb{R}^n \rtimes \mathrm{O}(n)$ be lattices. Then every isomorphism $\theta : \pi \to \pi'$ is a conjugation by an element of $\mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{R})$.*

THEOREM 3′. *Under each torus $\mathbb{Z}^n \backslash \mathbb{R}^n$, there are only finitely many flat manifolds which are covered by the torus.*

All three Bieberbach theorems have been generalized to the situation where G is a simply connected, connected nilpotent Lie group ([A1], [LR2], [DIM], [D2]). It is therefore natural to investigate the situation for solvable G. Unfortunately, as an example in this paper will show, the first Bieberbach theorem causes problems, even for the class of solvable Lie groups of type (R), a class of groups sharing many properties with the class of nilpotent Lie groups. A connected solvable Lie group G is called *of type* (R) (or *completely solvable*) if $\mathrm{ad}(X) : \mathfrak{G} \to \mathfrak{G}$ has only real eigenvalues for each $X \in \mathfrak{G}$. A connected solvable Lie group G is called *of type* (E) (or *exponential*) if $\exp : \mathfrak{G} \to G$ is surjective. Some important properties of such groups are listed below.

Although we will show that there are problems in generalizing the first Bieberbach Theorem, we are able to show which Lie groups of type (E) allow a straightforward generalization of the first Bieberbach theorem (Theorem 1'). In this respect, we need the following definition.

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