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SCHRÖDINGER FLOW FOR MAPS INTO KÄHLER MANIFOLDS*

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Abstract. In this paper, we show that there exists a global Schrödinger flow for a map from the unit circle S^1 into a Kähler manifold with constant holomorphic sectional curvature. Moreover, the Schrödinger flow obeys a conservative law. We also discuss the Schrödinger flow from a compact Riemann surface into a compact Kähler manifold and prove that the Cauchy problem admits a smooth local solution.

1. Introduction. Recently, in [DW], Ding and Wang considered Schrödinger flows for maps into symplectic manifolds. Let (N, ω, J, h) (sometimes denoted just by N, (N, J) or (N, J, h)) be a symplectic manifold equipped with a symplectic form ω, an almost complex structure J, and the Riemannian metric $h(\cdot, \cdot) = \omega(\cdot, J\cdot)$. Then, given a map u_0 from a Riemannian manifold (M, g) into (N, J), the Schrödinger flow $u(\cdot, t) : M \longrightarrow N$ for u_0 is defined by the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = J(u)\tau(u), \\ u(x, 0) = u_0(x), \end{cases} \tag{1.1}$$

where $\tau(u)$ is the tension field of u. In local coordinates,

$$\tau^\alpha(u) = \Delta u^\alpha + g^{ij}\Gamma^\alpha_{\beta\gamma}(u)\frac{\partial u^\beta}{\partial x^i}\frac{\partial u^\gamma}{\partial x^j},$$

where Δ is the Laplace-Beltrami operator on M with respect to the metric g and $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of the target manifold N.

For the case where $M = S^1$ and (N, J, h) is a Kähler manifold, Ding and Wang [DW] discussed existence and uniqueness for (1.1). They proved that it admits a unique local smooth solution if u_0 is smooth. Further, if (N, J, h) is a compact Kähler manifold with constant sectional curvature K, the solution is in fact global and smooth. This follows from the conservative law

$$\frac{d}{dt}\{\int_{S^1} |\tau(u)|^2\, dS - \frac{K}{4}\int_{S^1} |\nabla u|^4\, dS\} = 0.$$

We point out that if N is a compact Kähler manifold with constant sectional curvature, then it is either a closed surface or a flat complex torus of higher dimension.

We recall that the Heisenberg spin chain system (also called ferromagnetic spin chain system) is given by

$$\frac{\partial u}{\partial t} = u \times \Delta u,$$

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