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A FOCUS ON FOCAL SURFACES*

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Many classical problems in algebraic geometry have regained interest when techniques from differential geometry were introduced to study them. The modern foundations for this approach has been given by Griffiths and Harris in 6, who obtained in this way several classical and new results in algebraic geometry. More recently, this idea has been successfully followed by McCrory, Shifrin and Varley in 12 and 13 to study differential properties of hypersurfaces in $\mathbb{P}^3$ and $\mathbb{P}^4$. In fact these two papers have greatly influenced the present work.

In this spirit, the subject of this paper is the systematic study of focal surfaces of smooth congruences of lines in $\mathbb{P}^3$. This is indeed a clear example of a topic of differential nature in algebraic geometry. The study of such congruences has been very popular among classical algebraic geometers one century ago. Especially Fano has given many important contributions to this field. An essential ingredient in his work has been the focal surface of the congruence. This point of view has been retaken by modern algebraic geometers, such as Verra and Goldstein, and also by Ciliberto and Sernesi in higher dimension.

What we find amazing in the papers by the classics is how much information they were able to provide about the focal surface of the known examples of congruences, in particular about its singular locus (and more especially about fundamental points). They seemed to have in mind some numerical relations that they never formulated explicitly. And even nowadays such kind of relations would require deep modern techniques, like multiple-point theory, but also this powerful machinery is not a priori enough since some generality conditions need to be satisfied.

As a sample of this, the degree and class of the focal surface –the only invariants easy to compute– can be derived immediately from the Riemann-Hurwitz formula. However these invariants, even in the easiest examples (see Example 2.4 or Remarks after Corollary 4.7) seem to be wrong at a first glance. This is due to the existence of extra components of the focal surface or to the possibility that the focal surface counts with multiplicity, although this was never mentioned explicitly by the classics. Even in 5, these possibilities seem not to have been considered.

The starting point of this work was to understand how the classics predicted the number of fundamental points of a congruence. We only know of one formula in the literature involving this number, which is however wrong (see Example 1.15 and the remark afterwards). So our first goal was to use modern techniques in order to rigorously obtain some of the classical results on the topic. Specifically, by regarding the focal surface of a congruence as a scheme, we reobtain its invariants (degree, class, class of its hyperplane section, sectional genus, and degrees of the nodal and cuspidal curves) and give them a precise sense.

We also restrict our attention to congruences of bisecants to a curve, or flexes to a surface (since they are special cases in the work by Goldstein), or bitangents to a surface (since all the lines of a congruence are bitangent to the focal surface). In

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