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CANONICAL KÄHLER CLASSES*

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Abstract. On a compact complex manifold (M, J) of Kähler type, consider the functional defined by the square of the L^2-norm of the scalar curvature with domain the space of Kähler metrics of fixed total volume. The infimum of this functional over metrics that represent a given class defines the energy of the class. We find the Euler-Lagrange equation for critical classes of this energy, and compute its Hessian at a critical class. We use these results to conclude that there exist a unique critical class for any del Pezzo surface as well as for any manifold with negative first Chern class. It follows that if the first Chern class is positive and not a critical class of the energy, it cannot be represented by an Einstein metric. We also draw some conclusions on the generic case of a manifold without holomorphic vector fields.

1. Introduction. Let (M, J) be a compact complex manifold of Kähler type. We denote by $\mathfrak{M}$ the set of Kähler forms on (M, J), an open cone in the infinite-dimensional affine space of closed $(1,1)$-forms. We will identify the Kähler form ω in $\mathfrak{M}$ with the corresponding Kähler metric g. Then, if the complex dimension of (M, J) is n and $\omega \in \mathfrak{M}$, the associated volume form is $d\mu_\omega := \omega^n/n!$.

It this context, it is quite natural to analyze the functional given by the square of the L^2-norm of the scalar curvature of a metric,

$$\omega \stackrel{\Phi}{\to} \int_M s_\omega^2 d\mu_\omega \, ,$$

taking $\mathfrak{M}$ as its domain. However, this functional is homogeneous of degree $n-2$ and, in general, will not admit critical points without the specification of a suitable normalization for its domain. Calabi [2, 3] solved this problem by fixing a class $\Omega \in H^{1,1}(M, \mathbb{C}) \subset H^2(M, \mathbb{R})$ in the Kähler cone, and searching for critical metrics of the restriction of Φ to $\mathfrak{M}_\Omega = \{\omega \in \mathfrak{M} : \ [\omega] = \Omega\}$:

$$\begin{array}{rcl} \mathfrak{M}_\Omega & \stackrel{\Phi_\Omega}{\to} & \mathbb{R} \\ \omega & \mapsto & \displaystyle\int_M s_\omega^2 d\mu_\omega \, . \end{array} \tag{1.1}$$

All metrics in $\mathfrak{M}_\Omega$ have fixed volume $\Omega^n/n!$. Those which are critical points of Φ_Ω are called *extremal*, and their mere existence is far from being a settled issue to this date.

On the other hand, we may also consider the set $\mathfrak{M}_v$ of Kähler metrics of fixed total volume v, and define the functional

$$\begin{array}{rcl} \mathfrak{M}_v & \stackrel{\Phi}{\to} & \mathbb{R} \\ \omega & \mapsto & \displaystyle\int_M s_\omega^2 d\mu_\omega \, . \end{array} \tag{1.2}$$

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