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DEFORMATIONS OF SYMPLECTIC LIE ALGEBROIDS, DEFORMATIONS OF HOLOMORPHIC SYMPLECTIC STRUCTURES, AND INDEX THEOREMS*

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1. Introduction. A deformation quantization of a smooth manifold M is by definition a formal multiplication law on the space $C^\infty(M)[[\hbar]]$

$$f * g = fg + \sum_{k \geq 1} (i\hbar)^k P_k(f, g),$$

where $*$ is an associative $\hbar$-linear product satisfying

$$\frac{1}{i\hbar}(f * g - g * f) = \{f, g\} + O(\hbar).$$

One usually requires P_k to be local, i.e. bidifferential, expressions in f and g. It is also convenient to assume that

$$1 * f = f * 1 = f.$$

[2].

A powerful theorem of Kontsevich [24] states that there is a bijection between the set of isomorphism classes of deformations of $C^\infty(M)$ and the set of equivalence classes of formal Poisson structures (i.e., of formal series $\varpi = \sum_{k\geq 0} \hbar^k \varpi^k$ satisfying $[\varpi, \varpi] = 0$ where $\{,\ \}$ is the Schouten-Nijenhuis bracket). In this paper, we classify deformations for a class of Poisson structures for which explicit methods of Fedosov work. Using those methods, we prove an index theorem for this class of deformations.

In the case when M is a symplectic manifold it is known that deformation quantizations always exist [9] and are classified by the points of the homogeneous space $\frac{1}{i\hbar}\omega + H^2(M, \mathbb{C}[[\hbar]])$ [31], [8], [9]. A simple geometric construction of deformation quantization of a symplectic manifold was given in [13] by Fedosov.

Fedosov's methods are well suited for study of a more general class of Poisson manifolds. In this paper we generalize both them and the classification results of [31] to the Poisson structures associated with symplectic Lie algebroids (this generality was suggested to us by A. Weinstein who also independently carried out the construction of Fedosov connections in [40]).

A Lie algebroid is a vector bundle E over M whose sheaf of sections is a sheaf of Lie algebras and a morphism of bundles

$$\rho : E \to TM$$

satisfying

$$\rho[\xi, \eta] = [\rho(\xi), \rho(\eta)]$$

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