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CLASSIFICATION OF INDEFINITE HYPER-KÄHLER SYMMETRIC SPACES*

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Abstract. We classify indefinite simply connected hyper-Kähler symmetric spaces. Any such space without flat factor has commutative holonomy group and signature $(4m, 4m)$. We establish a natural 1-1 correspondence between simply connected hyper-Kähler symmetric spaces of dimension $8m$ and orbits of the group $\mathrm{GL}(m, \mathbb{H})$ on the space $(S^4\mathbb{C}^n)^\tau$ of homogeneous quartic polynomials S in $n = 2m$ complex variables satisfying the reality condition $S = \tau S$, where τ is the real structure induced by the quaternionic structure of $\mathbb{C}^{2m} = \mathbb{H}^m$. We define and classify also complex hyper-Kähler symmetric spaces. Such spaces without flat factor exist in any (complex) dimension divisible by 4.

1. Introduction. We recall that a pseudo-Riemannian manifold (M, g) is called a symmetric space if any point $x \in M$ is an isolated fixed point of an involutive isometry s_x (called central symmetry with centre x). Since the product of two central symmetries s_x and s_y with sufficiently close centres is a shift along the geodesic (xy), the group generated by central symmetries acts transitively on M and one can identify M with the quotient $M = G/K$, where G is the connected component of the isometry group $\mathrm{Isom}(M, g)$ and K is the stabilizer of a point $o \in M$.

A symmetric space $(M = G/K, g)$ is called Kähler (respectively, hyper-Kähler) if its holonomy group $\mathrm{Hol}(M, g)$ is a subgroup of the pseudo-unitary group $\mathrm{U}(p, q)$ (respectively, of the pseudo-symplectic group $\mathrm{Sp}(p, q) \subset \mathrm{SU}(2p, 2q)$). Any hyper-Kähler symmetric space is in particular a homogeneous hypercomplex manifold. Homogeneous hypercomplex manifolds of compact Lie groups were constructed by Ph. Spindel, A. Sevrin, W. Troost, A. Van Proeyen [SSTVP] and by D. Joyce [J] and homogeneous hypercomplex structures on solvable Lie groups by M.L. Barberis and I. Dotti-Miatello [BD].

The classification of simply connected symmetric spaces reduces to the classification of involutive automorphisms σ of a Lie algebra $\mathfrak{g}$, such that the adjoint representation $\mathrm{ad}_{\mathfrak{k}}|\mathfrak{m}$ preserves a pseudo-Euclidean scalar product g, where

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}, \quad \sigma|\mathfrak{k} = 1, \quad \sigma|\mathfrak{m} = -1,$$

is the eigenspace decomposition of the involution σ. Note that the eigenspace decomposition of an involutive automorphism is characterized by the conditions

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}.$$

Such a decomposition is called a symmetric decomposition.

In fact, for any pseudo-Riemannian symmetric space $M = G/K$ the conjugation with respect to the central symmetry s_o with centre $o = eK$ is an involutive automorphism of the Lie group G, which induces an involutive automorphism σ of its Lie algebra $\mathfrak{g}$. The pseudo-Riemannian metric of M induces a $\mathfrak{k}$-invariant scalar product on $\mathfrak{m} \cong T_oM$, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ is the symmetric decomposition defined by

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