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EXTENSION OF LIPSCHITZ MAPS INTO 3-MANIFOLDS*

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Abstract. We prove that the universal covering Y of a closed nonpositively curved 3-dimensional Riemannian manifold possesses the following Lipschitz extension property: there exists a constant $c \geq 1$ such that every λ-Lipschitz map $f : S \to Y$ defined on a subset S of an arbitrary metric space X has a $c\lambda$-Lipschitz extension $\overline{f} : X \to Y$.

1. Introduction. We say that a metric space Y has the *Lipschitz extension property (L)* if there exists a constant $c \geq 1$ such that every λ-Lipschitz map $f : S \to Y$ defined on an arbitrary subset S of some metric space X can be extended to a $c\lambda$-Lipschitz map $\overline{f} : X \to Y$.

Obviously, to have property (L) is a bilipschitz invariant of Y. One can prove that the Lipschitz extension property implies that Y is contactible. A space with property (L) also satisfies a quadratic isoperimetric inequality for closed curves, i.e. a closed curve of length l in Y can be spanned by a surface with area $\leq c'l^2$. This follows from the fact that the arclength parametrization of the curve defined on the circle $S^1_r \subset \mathbb{R}^2$ of radius $r = \frac{l}{2\pi}$ is Lipschitz and can be extended to a Lipschitz map defined on the disc.

A classical result of McShane [M] states that $\mathbb{R}$ has the property (L) with constant $c(\mathbb{R}) = 1$. Applying this result to the coordinate functions, $\mathbb{R}^n$ has property (L) with constant $c(\mathbb{R}^n) = \sqrt{n}$. Lang [L] showed that the optimal constant for $\mathbb{R}^n$ has to depend on n and that (L) is not valid for an infinite-dimensional Hilbert space.

In [LPS] it is proved that the following three classes of Hadamard spaces have the property (L)

(1) the 2-dimensional Hadamard manifolds;
(2) the class of Gromov-hyperbolic Hadamard manifolds whose curvature is bounded by $-b^2 \leq K \leq 0$;
(3) the class of homogeneous Hadamard manifolds and euclidean Tits buildings.

In this paper we study the validity of (L) for 3-dimensional spaces. Let us first investigate the standard homogeneous 3-dimensional geometries:
S^3, $\mathbb{R}^3$, H^3, $S^2 \times \mathbb{R}$, $\mathrm{H}^2 \times \mathbb{R}$, NIL, SOL, $\widetilde{PSL}_2(\mathbb{R})$.

Since S^3 and $S^2 \times \mathbb{R}$ are homotopically nontrivial, they do not satisfy the extension property, while $\mathbb{R}^3$, H^3 and $\mathrm{H}^2 \times \mathbb{R}$ satisfies (L) by McShane's result and the case (3) above. Since $\widetilde{PSL}_2(\mathbb{R})$ is bilipschitz to $\mathrm{H}^2 \times \mathbb{R}$ (this observation is due to Epstein, Mess and Gersten according to [KLe]) the property (L) is satisfied. On the other hand NIL and SOL do not share (L) since they do not allow quadratic isoperimetric inequalities (see [Eetal]).

We show that the property (L) holds for a large class of simply connected 3-manifolds. Our results can be summarized by the following two theorems.

THEOREM A. Let Y be the universal covering of a nonpositively curved, closed Riemannian 3-manifold. Then Y satisfies (L).

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