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BIELLIPTIC DRINFELD MODULAR CURVES*

ANDREAS SCHWEIZER†

Abstract. We determine all Drinfeld modular curves $X_0(\mathfrak{n})$ that are bielliptic, i.e. double covers of elliptic curves. This allows us to find all $X_0(\mathfrak{n})$ that have infinitely many quadratic points over $\mathbb{F}_q(T)$. We also improve one of the results of Poonen concerning uniform boundedness for Drinfeld modules of rank 2.

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0. Introduction. Let $\mathfrak{n} \in \mathbb{F}_q[T]$. Then the Drinfeld modular curve $X_0(\mathfrak{n})$ has infinitely many rational points over $\mathbb{F}_q(T)$ if and only if it is a rational curve, which is the case if and only if $deg(\mathfrak{n}) \leq 2$. This results from a criterion in [Sa], since the curves $X_0(\mathfrak{n})$ with positive genus are non-isotrivial and the elliptic ones have rank 0.

The question which of the curves $X_0(\mathfrak{n})$ have infinitely many quadratic points over $\mathbb{F}_q(T)$, or more generally, over a finite extension of $\mathbb{F}_q(T)$, is related in a more complicated way to the problem which $X_0(\mathfrak{n})$ are hyperelliptic or bielliptic. The hyperelliptic ones have been determined in [Sch3] and now we treat the bielliptic case.

In the classical situation (i.e. over $\mathbb{C}$ or $\mathbb{Q}$) all bielliptic modular curves $X_0(n)$ have been determined in [Ba]. In order to reduce the possibly bielliptic modular curves to a finite number one uses a strategy originally due to A. Ogg, namely counting points on a suitable reduction. This works equally well in the classical and in the Drinfeld setting. However, when dealing with the remaining cases, life is harder for us, since, unlike the classical situation, not much is known about the full automorphism group of $X_0(\mathfrak{n})$. Therefore we have to use several different strategies and ad hoc arguments.

The first main result (Theorem 4.6) is that up to affine tranformations of $\mathfrak{n}$ there are exactly 12 bielliptic curves $X_0(\mathfrak{n})$ (of which 6 are at the same time hyperelliptic). All bielliptic involutions we find are modular automorphisms (and often even Atkin-Lehner involutions). However, we do not claim that these 12 curves have no other bielliptic involutions, although this seems unlikely.

Section 5 contains the other main results: Theorem 5.1 relates the two properties, to have infinitely many quadratic points and being bielliptic. Theorem 5.3 lists all $X_0(\mathfrak{n})$ with infinitely many quadratic points over $\mathbb{F}_q(T)$. As an application we improve a result from [Po] on the uniform boundedness of the $\mathfrak{p}$-primary torsion of $\mathbb{F}_q[T]$-Drinfeld modules of rank 2 over a fixed finite extension L of $\mathbb{F}_q(T)$; namely, in Theorem 5.4 we make the bound uniform over all quadratic extensions of L. See also Remark 5.5, where further possible generalizations are indicated.

1. Basic Facts. As always, $\mathbb{F}_q$ is the finite field with q elements. Throughout this paper K stands for the rational function field $\mathbb{F}_q(T)$ and A for the polynomial ring $\mathbb{F}_q[T]$, where sometimes q is specified in the context. The monic irreducible elements of A are called primes and denoted by the letter $\mathfrak{p}$. We refer to the corresponding places of K as finite places and use the symbol ∞ for the place with uniformizer $\frac{1}{T}$.

For every non-zero $\mathfrak{n} \in A$ we have the Drinfeld modular curve $X_0(\mathfrak{n})$. Here we are interested in these curves as algebraic curves over K. We refer to [Sch3] for a

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†Korea Institute for Advanced Study (KIAS), 207-43 Cheongryangri-dong, Dongdaemun-gu, Seoul 130-012, Korea (schweiz@kias.re.kr).