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008

LOW INDEX MINIMAL HYPERSURFACES OF SPHERES*

OSCAR PERDOMO†

Abstract. Let M be a compact oriented non-equatorial minimal hypersurface of the unit n-dimensional sphere. Suppose that for any non-zero vector in $w \in \mathbf{R}^{n+1}$ there exists an orthogonal matrix B such that $B(M) = M$ and $B(w) \neq w$. Since all known examples of minimal hypersurfaces have antipodal symmetry, they satisfy this condition.

We prove that: i) the stability index of M is greater than or equal to $n+2$ with strict inequality, unless M is a Clifford hypersurface; ii) the difference between the first two eigenvalues of the Jacobi operator is less than or equal to $n-1$ with strict inequality, unless the norm of the second fundamental form is constant; and iii) if M has antipodal symmetry and is not a Clifford hypersurface then the index is greater than $n+3$. Moreover if the unit normal vector is even, the index is greater than $2n+2$.

1. Introduction. Let M be a compact, oriented minimal hypersurface immersed in the n-dimensional sphere S^n. Let ν be a unit normal vector field along M. For any tangent vector $v \in T_mM$, $m \in M$, the shape operator A is given by $A(v) = -\bar{\nabla}_v \nu$, where $\bar{\nabla}$ denotes the Levi Civita connection in S^n. We will denote by Δ the Laplacian on M. Given any function $f : M \longrightarrow \mathbf{R}^1$ we can form the 1-parameter variational family defined by

$$M_t = \{\exp(m, tf(m)\nu) : m \in M\}$$

where $\exp(m, \cdot)$ is the exponential map at $m \in S^n$.

It is well known (see e.g. [9]) that the area of M_t satisfies:

$$\frac{d}{dt}(\text{area}(M_t))\big|_{t=0} \quad = \quad 0 \qquad \text{(minimality of } M\text{)}$$

$$\frac{d^2}{dt^2}(\text{area}(M_t))\big|_{t=o} \quad = \quad \int_M J(f)f \qquad \text{(second variation formula)}$$

where J is the Jacobi operator on M, given by

$$J = -\Delta - \|A\|^2 - (n-1)$$

When $\int_M J(f)f < 0$ for some $f \in C^\infty(M)$, we see that area(M) >area(M_t) for small values of $t \neq 0$. In particular this implies that the minimal hypersurface, M, while a critical point of the area functional, is not a local minimum. For hypersurfaces in spheres this is always the case, since we can take $f \equiv 1$.

The Jacobi operator induces a symmetric bilinear form

$$I(f, g) = \int_M J(f)g$$

known as *the index form* on M. Let V be a subspace of $C^\infty(M)$ on which the index form is negative definite, i.e. $\int_M J(f)f < 0$ for all $f \in V$. Since the eigenvalues

*Received February 17, 2001; accepted for publication March 6, 2001.

†Indiana University, Bloomington IN 47405, USA (osperdom@mafalda.univalle.edu.co).