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THE INTEGRAL ON QUANTUM SUPERGROUPS OF TYPE $A_{R|S}$*

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Abstract. Quantum groups of type $A_{r|s}$ generalize the general linear supergroups $GL(r|s)$. We compute the integral on these quantum supergroups and whence derive a quantum analogue of (super) HCIZ integral formula.

Introduction. A quantum supergroup of type $A_{r|s}$ is a Hopf superalgebra, which, as an algebra, is a quotient of a free non-commutative tensor algebra on a certain finite dimensional vector superspace V, by certain relations, determined in terms of a Hecke symmetry of birank $(r+1, s+1)$ acting on $V \otimes V$. If the Hecke symmetry reduces to the ordinary super-permuting operator in the category of vector superspaces, we recover the quantum general linear supergroup $GL(r+1, s+1)$, (Section 1).

In this work, we compute the integral on quantum supergroups of type $A_{r|s}$. The notion of integral on Hopf algebras was introduced by Sweedler [24]. This is an analogue of the notion of invariant integrals on compact groups (see, e.g., [21]). In fact, the algebra of representative functions on a compact group is dense in the algebra of all continuous complex valued functions by Peter-Weyl's theorem. The former algebra is a Hopf algebra and the definition of the invariant integral can be given in a purely algebraic way using the coalgebra structure, namely, a (left) integral on a Hopf algebra H, defined over a field k, is a linear form $\int : H \longrightarrow k$, satisfying the following condition:

$$\int(a) = m(\int \otimes \mathsf{id}_H)\Delta(a),$$

where m, Δ denote the product and coproduct on H and we identify the field k with a subspace of H by means of the unit of H. This definition can also be extended for Hopf superalgebras. On the other hand, the existence of integral on compact supergroups was established by Berezin [3]. Unlike the case of compact group, the volume of a compact supergroup may vanish, as it happens to the supergroups $U(r|s)$ of unitary supermatrices.

A formula for the integral on the group $U(d)$ was obtained by Itzykson and Zuber [4]. This formula computes the integral at the function of the form $\mathsf{tr}_{M,N,n}(U) := \left[\mathsf{tr}(MUNU^{-1})\right]^n$, where M, N are hermitian matrices. It can be given in the following form

$$\int_{U(d)} \left[\mathsf{tr}(MUNU^{-1})\right]^n [dU] = \sum_{\lambda \in \mathcal{P}_n^d} \frac{d_\lambda}{r_\lambda} \Phi_\lambda(M)\Phi_\lambda(N) \tag{0.1}$$

for any hermitian matrices M and N. Here, $\mathcal{P}_n^d$ is the set of partitions of n of length d. Φ_λ is the irreducible character of $U(d)$, corresponding to the partition λ. If $\xi_1, \xi_2, ..., \xi_d$

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