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CONFORMAL LOWER BOUNDS FOR THE DIRAC OPERATOR OF EMBEDDED HYPERSURFACES*

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Abstract. We find sharp lower bounds for the first nonnegative eigenvalue of the classical intrinsic Dirac operator of a compact hypersurface bounding a domain in a Riemannian spin manifold. These estimates are given in terms of scalar (spectral) conformal invariants of the enclosed domain which are involved in the solution of the Yamabe problem.

1. Introduction. In [Hij1], the first author used the conformal behavior of the nonzero eigenvalues λ of the Dirac operator to show that on a closed connected spin manifold λ^2 is, up to a dimensional constant, at least equal to the first eigenvalue of the Yamabe operator.

Using techniques for pseudo-differential operators and Sobolev embeddings, J. Lott [Lo] proved the existence of conformal lower bounds for λ^2. In [Hij2] it is shown that the Yamabe number, i.e., the infimum over a conformal class of metrics of the normalized total scalar curvatures, gives such a lower bound.

In [HMZ1], the present authors considered a domain Ω with boundary Σ inside a compact $(n+1)$-dimensional spin manifold M and showed that if M has nonnegative scalar curvature, then the first nonnegative eigenvalue λ_1 of the intrinsic Dirac operator of Σ satisfies

$$\lambda_1 \geq \frac{n}{2} \inf_{\Sigma} H. \tag{1.1}$$

where H is the mean curvature. The main new ingredient was to use the boundary condition of Atiyah-Patodi-Singer (APS) type for Dirac operators. If the ambient space has nonnegative Einstein tensor, (1.1) improves Friedrich's inequality for embedded hypersurfaces. As an application, a spinorial proof of the classical Alexandrov Theorem was obtained.

The present paper is devoted to the conformal aspect of the results obtained in [HMZ1]. We improve (1.1) by showing that

$$\lambda_1 \geq \frac{n}{2} \, \nu_1(B) \tag{1.2}$$

where $\nu_1(B)$ is the first eigenvalue of the conformal mean curvature operator B (see Theorem 9 for a precise statement). Here we don't need to assume that M has nonnegative scalar curvature. Furthermore, the limitting-case of (1.2) is characterized by the existence of a parallel spinor on the ambient manifold for a metric in the conformal class. We then use the Hölder inequality to show that

$$\lambda_1 \geq \frac{n}{2} \frac{\mathcal{Q}(\Omega, \Sigma)}{\mathrm{vol}\,(\Sigma)^{\frac{1}{n}}} \tag{1.3}$$

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