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RESIDUES OF THE BOTT CLASS AND AN APPLICATION TO THE FUTAKI INVARIANT*

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Abstract. The Bott class for transversely holomorphic foliations and the Futaki invariant for complex manifolds are discussed. The main tools are a generalization of Heitsch's residue and the integration of Čech-de Rham cochains introduced by Lehmann. It is shown that the Bott class admits a natural decomposition as a differential character in the sense of Cheeger and Simons. The decomposition implies that the real and the complex parts of the Futaki invariant have a certain difference from the viewpoint of dynamical systems.

Introduction. In the theory of characteristic classes, so-called Chern-Simons classes are very important. In this paper, the Futaki invariant and more generally, the Bott class of transversely holomorphic foliations are studied. The Bott class is the most fundamental characteristic class for transversely holomorphic foliations. Here a foliation is said to be transversely holomorphic if its holonomy is written in terms of biholomorphic diffeomorphisms. The Bott class is a Chern-Simons class or Cheeger-Simons class for the complex normal bundle of the foliation. Under an additional assumption on the complex normal bundle, the definition of the Bott class becomes completely parallel to that of the Godbillon-Vey class. In this sense, the Bott class is a counterpart of the Godbillon-Vey class in the complex category. However, its properties are not yet well-understood. One of difficulties is that the Bott class is an element of $\boldsymbol{C}/\boldsymbol{Z}$-coefficient cohomology class. Indeed, the imaginary part can be written by a closed form and is easier to study, while the study of the real part often needs some additional techniques or notions, e.g. η-invariants. In this paper, first we introduce a modified version of Čech-de Rham complex and show that a certain representative of the Bott class can be treated as a differential form (Definition 2.13 and Theorem 2.19). This is based on the Chern-Weil theory on the Čech-de Rham complex found in [23]. There have been already some approaches to consider the de Rham cohomology with coefficients other than $\boldsymbol{C}$ or $\boldsymbol{R}$ [11], [21], where simplicial complexes are used. Under our formulation, the integration of Čech-de Rham cochains introduced by Lehmann [19] plays an essential role and it allows the construction to be elementary.

One of the main results is a generalization of Heitsch's residue of the Bott class (Theorem 3.4). Combined with a structure theorem of complex codimension-one foliations due to Ghys–Gomez-Mont–Saludes, this residue gives a generalization of Duminy's theorem in a weak form (Corollary 5.4). As an application of the residue to higher codimensional foliations, the Futaki invariant is discussed. It is a group character defined on the groups of biholomorphic automorphisms of a complex manifold and is valued in $\boldsymbol{C}/\boldsymbol{Z}$. Given an automorphism σ of a complex manifold M, one can naturally construct a foliated manifold M_σ called the suspension, and the Futaki invariant is essentially the Bott class of this foliation. Calculations of the Futaki invariant often involve the η-invariant, which need some additional structures. The calculations

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