COMMUNICATIONS IN INFORMATION AND SYSTEMS
Vol. 1, No. 1, pp. 101-112, January 2001
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OPTIMAL APPROXIMATION OF FUNCTIONS*

YISHAO ZHOU†, MAGNUS EGERSTEDT‡, AND CLYDE MARTIN§

Abstract. In this paper a complete theory of $L^2[0,T]$ approximation for generalized interpolating and smoothing splines is established. It is shown that under general conditions smoothing splines both smooth noisy data and recover the underlying function in the sense of $L^2[0,T]$ convergence. It is shown that generalized interpolating splines converge to the underlying function in $L^2[0,T]$.

1. Introduction. In a series of papers by Egerstedt, Martin and Sun, [11, 12, 13, 14, 16] some of the basic properties of smoothing splines have been established. However the basic question of approximation for such splines has not been answered and neither has the same question for generalized interpolating splines been answered, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 15, 18, 19]. For polynomial splines, both smoothing and interpolating, there is a considerable literature on approximation properties. For polynomial interpolating splines any good graduate numerical analysis text will provide ample references. For polynomial smoothing splines the seminal work of Grace Wahba, [17], provides the best reference.

In this paper we will consider five problems and their interrelations. Let a data set V_ϵ be given

$$V_\epsilon = \{(t_i, f(t_i) - \epsilon_i) : \quad i = 1, \cdots, N\}$$

and let

$$V_0 = \{(t_i, f(t_i)) : \quad i = 1, \cdots, N\}.$$

Let the system

$$\dot{x} = Ax + bu \tag{1.1}$$

$$y = cx \tag{1.2}$$

be controllable and observable. Furthermore, let

$$cb = cAb = ca^2b = \cdots = cA^{n-2}b = 0. \tag{1.3}$$

The first problem we consider is just the existence of generalized interpolating splines.

Remark: *This condition is not natural from the point of view of control theory. However, from the point of view of approximation and from the point of view of*

*Invited paper; received May 12, 2000; accepted for publication June 14, 2000. Yishao Zhou is supported in part by Swedish Natural Science Research Council (NFR). The work by Magnus Egerstedt is supported in part by the US Army Research Office, Grant number DAAG 5597-1-0114, and in part by the Sweden-America Foundation 2000 Research Grant. Clyde Martin is supported in part by NSF Grants ECS 9720357 and ECS 9705312 and gratefully acknowledges the support of KTH in the spring of 2000.

† Department of Mathematics, University of Stockholm, Stockholm Sweden
‡ Division of Engineering and Applied Science, Harvard University, Cambridge, MA, USA
§ Department of Mathematics and Statistics, Texas Tech University, Lubbock, Texas, USA

trajectory planning it is an entirely natural condition. We treat $y(t)$ as a position and if we write the system in terms of $y(t)$ and its derivatives we have a system that satisfies the conditions of the assumption.

PROBLEM 1. *Let*

$$L^N(u) = \int_0^T u^2(t)dt \tag{1.4}$$

and for each point in V_0 let a linear equation be given:

$$f(t_i) = y(t_i) = \int_0^{t_i} ce^{A(t-s)}bu(s)ds, \quad i = 1, \cdots, N. \tag{1.5}$$

Find a control $u_0^N(t) \in L^2[0,T]$ such that u_0^N is the solution of

$$\min_u L_0^N(u) \tag{1.6}$$

subject to the constraints of Equation 1.5.

The second problem is the problem of smoothing splines for noisy data, i.e., the data set V_ϵ.

PROBLEM 2. *Let a cost function $J_\epsilon^N(u)$ be given as*

$$J_\epsilon^N(u) = \sum_{i=1}^N w_{iN}(y(t_i) - f(t_i) - \epsilon_i)^2 + \lambda \int_0^T u^2(t)dt. \tag{1.7}$$

Find $u_{\epsilon,\lambda}^N$ that is the solution of

$$\min_u J_\epsilon^N(u). \tag{1.8}$$

The third problem that we consider is the problem of smoothing splines with deterministic data.

PROBLEM 3. *Let a cost function $J^N(u)$ be given as*

$$J^N(u) = \sum_{i=1}^N w_{iN}(y(t_i) - f(t_i))^2 + \lambda \int_0^T u^2(t)dt. \tag{1.9}$$

Find u_λ^N that is the solution of

$$\min_u J^N(u). \tag{1.10}$$

The fourth problem is a problem of tracking in continuous time.

PROBLEM 4. *Let a cost function $J_\lambda(u)$ be given as*

$$J_\lambda(u) = \int_0^T (y(t) - f(t))^2 + \lambda u^2(t)dt. \tag{1.11}$$

Find u_λ that is the solution of

$$\min_u J_\lambda(u). \tag{1.12}$$

The fifth problem is a problem of approximation.

PROBLEM 5. *Let a cost function $J(u)$ be given as*

$$J(u) = \int_0^T (y(t) - f(t))^2 dt. \tag{1.13}$$

Find u^* that is the solution of

$$\min_u J(u). \tag{1.14}$$

Remark: *We prove that the controls in the five problems converge in various senses. Because the linear operator that maps the control to the output (the generalized spline function) is a continuous operator the generalized splines likewise converge.*

We must construct the solution to Problem 4 which we will do in the next section. In Section 3 we will prove the existence of an approximate solution to Problem 5. In the final section we outline, based on the results of this paper and the results of [13, 14, 16, 18] the interrelations between the five problems.

2. Derivation of the Basic Approximation of Problem 4. The solutions to problems 1, 2 and 3 are well established in [18, 16, 14]. In [16] a brief description of the solution to Problem 4 is given. In this section we will give a rather complete analysis of the solution to Problem 4. Let the cost function of Problem 4

$$J(u) = \int_0^T (cx(t) - f(t))^2 + \lambda u^2(t) dt, \tag{2.1}$$

be given. We define the linear constraint to be the minimal system

$$\sum : \quad \begin{array}{l} \dot{x} = Ax + bu, \quad x(0) = x_0 \\ \\ y = cx. \end{array} \tag{2.2}$$

The problem we attack is simple: we will show in Section 3 that as λ goes to zero the optimal function $cx(t)$ approaches $f(t)$ in the L_2 norm for appropriate choice of the initial data x_0. The first problem is the following which is just a restatement of Problem 4.

Problem: *Minimize J(u) subject to the constraint of $\sum$.*

We first construct the Hamiltonian

$$H(x, u, p) = \frac{1}{2}(cx(t) - f(t))^2 + \frac{\lambda}{2}u^2(t) + (Ax + bu)^{\perp} p(t). \tag{2.3}$$

From standard theory we must solve the following equations.

$$\frac{\partial}{\partial x} H(x, u, p) = -\dot{p} \tag{2.4}$$

$$\frac{\partial}{\partial u} H(x, u, p) = 0 \tag{2.5}$$

$$\frac{\partial}{\partial p} H(x, u, p) = \dot{x}^{\perp}. \tag{2.6}$$

Calculating the partial derivatives we have

$$\begin{aligned}\frac{\partial}{\partial x}H(x,u,p) &= -\dot{p}\\ &= (cx(t)-f(t))c^{\perp}+A^{\perp}p\end{aligned}$$

giving us

$$\dot{p}(t) = -c^{\perp}cx(t) - A^{\perp}p(t) + c^{\perp}f(t). \tag{2.7}$$

Calculating the next we have

$$\begin{aligned}\frac{\partial}{\partial u}H(x,u,p) &= 0\\ &= \lambda u(t) + b^{\perp}p(t)\end{aligned}$$

giving us

$$u(t) = -\frac{1}{\lambda}b^{\perp}p(t). \tag{2.8}$$

Calculating the last partial derivative we have

$$\begin{aligned}\frac{\partial}{\partial p}H(x,u,p) &= \dot{x}^{\perp}\\ &= (Ax(t)+bu(t))^{\perp}\end{aligned}$$

giving us

$$\dot{x} = Ax + bu. \tag{2.9}$$

Writing these in matrix form we have

$$\frac{d}{dt}\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} A & \frac{-1}{\lambda}bb^{\perp} \\ -c^{\perp}c & -A^{\perp} \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ c^{\perp} \end{pmatrix} f(t) \tag{2.10}$$

with

$$x(0) = x_0 \quad p(T) = 0. \tag{2.11}$$

We now introduce the Riccati equation. We define the transformation

$$\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} I & 0 \\ R(t) & I \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix}. \tag{2.12}$$

Applying the transformation to equation 2.10 we have

$$\frac{d}{dt}\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} A & \frac{-1}{\lambda}bb^{\perp} \\ -c^{\perp}c & -A^{\perp} \end{pmatrix}\begin{pmatrix} I & 0 \\ R(t) & I \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ c^{\perp} \end{pmatrix} f(t)$$

$$= \begin{pmatrix} A - \frac{1}{\lambda}bb^{\perp}R(t) & -\frac{1}{\lambda}bb^{\perp} \\ -c^{\perp}c - A^{\perp}R(t) & -A^{\perp} \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ c^{\perp} \end{pmatrix} f(t)$$

$$= \begin{pmatrix} 0 & 0 \\ \dot{R}(t) & 0 \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} I & 0 \\ R(t) & I \end{pmatrix}\frac{d}{dt}\begin{pmatrix} x \\ w \end{pmatrix}.$$

Solving we have

$$\frac{d}{dt}\begin{pmatrix} x \\ w \end{pmatrix}$$
$$= \begin{pmatrix} A - \frac{1}{\lambda}bb^{\perp}R(t) & -\frac{1}{\lambda}bb^{\perp} \\ Z(t) & -(A - \frac{1}{\lambda}bb^{\perp}R(t))^{\perp} \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ c^{\perp} \end{pmatrix} f(t)$$

where

$$Z(t) = -\dot{R}(t) - R(t)[A - \frac{1}{\lambda}bb^{\perp}R(t)] - c^{\perp}c - A^{\perp}R(t).$$

Setting $Z(t) = 0$ we have the decoupled equations

$$\frac{d}{dt}\begin{pmatrix} x \\ w \end{pmatrix} = \begin{pmatrix} A - \frac{1}{\lambda}bb^{\perp}R(t) & -\frac{1}{\lambda}bb^{\perp} \\ 0 & -(A - \frac{1}{\lambda}bb^{\perp}R(t))^{\perp} \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix}$$
$$+ \begin{pmatrix} 0 \\ c^{\perp} \end{pmatrix} f(t) \tag{2.13}$$

and the Riccati equation

$$\dot{R}(t) = -R(t)[A - \frac{1}{\lambda}bb^{\perp}R(t)] - c^{\perp}c - A^{\perp}R(t) \tag{2.14}$$

Since we have that $w(t) = -R(t)x(t) + p(t)$ a natural choice of initial data for w and R is

$$w(T) = 0 \text{ and } R(T) = 0. \tag{2.15}$$

This does present some complications for the solution since now the systems are time varying. Under the conditions of minimality that we have imposed we can solve the Riccati equation on the interval $[0, T]$. Now let $\phi(t)$ be the semi-group associated with the equation

$$\dot{z} = (A - \frac{1}{\lambda}bb^{\perp}R(t))z$$

so that we have

$$z(t) = \phi(t)z(0)$$

and let

$$\phi(t,s) = \phi(t)\phi^{-1}(s).$$

We then can solve for w as

$$w(t) = \int_t^T \phi(s,t)^\perp c^\perp f(s)ds \tag{2.16}$$

and then solving for $x(t)$ we have

$$x(t) = \phi(t)x_0 - \frac{1}{\lambda}\int_0^t \phi(t,s)bb^\perp \int_s^T \phi(r,s)^\perp c^\perp f(r)drds. \tag{2.17}$$

We can also replace $R(t)$ by a constant matrix, the positive definite solution of the algebraic Riccati equation (ARE) but at the expense of some further calculation.

Since R is a solution of the algebraic Riccati equation we have that $Z(t) = 0$ and we have, similar to equation 2.13,

$$\frac{d}{dt}\begin{pmatrix} x \\ w \end{pmatrix} = \begin{pmatrix} A - \frac{1}{\lambda}bb^\perp R & -\frac{1}{\lambda}bb^\perp R \\ 0 & -(A - \frac{1}{\lambda}bb^\perp)^\perp \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ c^\perp \end{pmatrix} f(t). \tag{2.18}$$

Now the initial and terminal data are more problematic. We have

$$w(t) = -Rx(t) + p(t)$$

and we don't know either $p(0)$ or $x(T)$. We make the choice to assume that we can calculate $w(T)$ and use it as dummy variable for the time being.

Let

$$\hat{A} = A - \frac{1}{\lambda}bb^\perp R.$$

We then have by solving for $w(t)$

$$w(t) = e^{\hat{A}^\perp (T-t}w(T) - \int_t^T e^{\hat{A}^\perp (t-s)}c^\perp f(s)ds \tag{2.19}$$

keeping in mind that $w(T)$ is not known at this point. We solve for $x(t)$ to obtain

$$\begin{aligned} x(t) = {} & e^{\hat{A}t}x_0 - \\ & \frac{1}{\lambda}\int_0^t e^{\hat{A}(t-s)}bb^\perp\left(e^{\hat{A}^\perp (T-s)}w(T) - \int_s^T e^{\hat{A}^\perp (s-r)}c^\perp f(r)dr\right)ds. \end{aligned} \tag{2.20}$$

Using the fact that

$$w(T) = -Rx(T) + p(T) = -Rx(T)$$

and evaluating $x(T)$ we have

$$\begin{aligned}-w(T) &= Rx(T)\\ &= Re^{\hat{A}T}x_0 - R\frac{1}{\lambda}\int_0^T e^{\hat{A}(T-s)}bb^{\perp}e^{\hat{A}^{\perp}(T-s)}w(T)ds\\ &\quad -R\frac{1}{\lambda}\int_0^T e^{\hat{A}(T-s)}bb^{\perp}\int_s^T e^{\hat{A}^{\perp}(s-r)}c^{\perp}f(r)drds.\end{aligned}$$

Rearranging the equations we have

$$\left(-I + R\frac{1}{\lambda}\int_0^T e^{\hat{A}(T-s)}bb^{\perp}e^{\hat{A}^{\perp}(T-s)}ds\right)w(T) =$$

$$Re^{\hat{A}T}x_0 - R\frac{1}{\lambda}\int_0^T e^{\hat{A}(T-s)}bb^{\perp}\int_s^T e^{\hat{A}^{\perp}(s-r)}c^{\perp}f(r)drds$$

and hence

$$\begin{aligned}w(T) = &\left(-I + R\frac{1}{\lambda}\int_0^T e^{\hat{A}(T-s)}bb^{\perp}e^{\hat{A}^{\perp}(T-s)}ds\right)^{-1}\times\\ &\left(Re^{\hat{A}T}x_0 - R\frac{1}{\lambda}\int_0^T e^{\hat{A}(T-s)}bb^{\perp}\int_s^T e^{\hat{A}^{\perp}(s-r)}c^{\perp}f(r)drds\right)\end{aligned} \tag{2.21}$$

Substituting this into equation (2.20) we then have an expression for $cx(t)$ and hence an approximation to the function $f(t)$.

3. An Approximation to Problem 5. In this section we will show that for the cost functions of Problem 4 and 5,

$$J_\lambda(u) = \int_0^T (y(t) - f(t))^2 + \lambda u^2(t)dt,$$

that for $\lambda = 0$ that the minimum value of the cost function is 0. We show this by proving that under suitable conditions on the system the image of the operator

$$H(u) = \int_0^t ce^{A(t-s)}bu(s)ds \tag{3.1}$$

is dense.

We state the theorem.

THEOREM 3.1. *Let the controllable, observable system*

$$\begin{aligned}\dot{x} &= Ax + bu, \quad x \in R^n, \quad u \in R\\ y &= cx, \quad y \in R\end{aligned}$$

have the property that 1) $cb = cAb = cA^2b = \cdots = cA^{n-2}b = 0$ *and 2) the* $\text{spec(A)} \subset C^-$*. Under these conditions*

$$H : L^2[a,b] \to L^2[a,b]$$

and the image of H *is dense in* $L^2[a,b]$ *for all choices of* $0 \le a < b \le \infty$.

Proof: Without loss of generality we let $a = 0$.

$$V = \{ \int_0^t ce^{A(t-s)}bu(s)ds : \quad u \in L^2[0,b]\}.$$

Now V is a linear subspace which may or may not be closed. We will prove that the closure of V is dense in $L^2[0,b]$ We consider here two cases. If $b = \infty$ then a classical theorem that H maps square integrable functions to square integrable functions holds and we are finished. For the case when $b < \infty$ we can use the classical result and embed $L^2[0,b]$ in $L^2[0,\infty]$ by extending each function to be zero outside of the interval $[0,b]$.

Suppose V is not dense. Then there is a linear subspace not contained in the closure of V and hence there is a function $f \in L^2[0,b]$ that is orthogonal to V. We calculate this function. For all $u \in L^2[0,b]$ we have

$$0 = \int_0^b f(t) \int_0^t ce^{A(t-s)}bu(s)dsdt \tag{3.2}$$

$$= \int_0^b \int_s^b f(t)ce^{A(t-s)}bu(s)dtds \tag{3.3}$$

and hence we have that

$$\int_s^b f(t)ce^{A(t-s)}bdt = 0 \tag{3.4}$$

We rewrite this as

$$ce^{-As} \int_s^b e^{At}bf(t)dt = 0.$$

Since this is identically zero as a function of s we can differentiate it to obtain

$$0 = -cAe^{-As} \int_s^b e^{At}bf(t)dt - ce^{-As}e^{As}bf(s) \tag{3.5}$$

$$= -cAe^{-As} \int_s^b e^{At}f(t)dt. \tag{3.6}$$

Continuing in this manner we obtain

$$-cA^{n-2}e^{-As} \int_s^b e^{At}bf(t)dt = 0. \tag{3.7}$$

By differentiating this and using the fact that $cA^{n-2}b = 0$ we obtain

$$-cA^{n-1}e^{-As} \int_s^b e^{At}bf(t)dt = 0. \tag{3.8}$$

We now use the fact that the system is minimal to arrive at

$$e^{-As}\int_s^b e^{At}bf(t)dt = 0 \tag{3.9}$$

and that

$$\int_s^b e^{At}bf(t)dt = 0. \tag{3.10}$$

Differentiating this with respect to s we have $e^{As}bf(s) = 0$ and since b is injective $f(s) \equiv 0$.

We have thus shown that there cannot be a nonzero vector orthogonal to V and hence V is dense in $L^2[0, b]$ and the theorem is proved.

This establishes the following result.

THEOREM 3.2. *Let $f(t) \in L^2[0, T]$. There exists an optimal control u that minimizes the cost function of Problem 5 if and only if f is in the image of H. However there exists a sequence of controls such that the limiting value of the cost function of Problem 5 is 0. Problem 4 generates such a sequence as function of λ and Problem 1 generates such a sequence as a function of N.*

4. Interrelations. We begin with a simple theorem that shows the relationship of Problem 3 to Problem 1.

THEOREM 4.1. *Let u_0^N be the solution of Problem 1 and we let N be fixed. Let u_λ^N be the solution of Problem 3 and again let N be fixed. Then*

$$\lim_{\lambda\to 0} u_\lambda^N = u_0^N \tag{4.1}$$

Proof: We first note that

$$J^N(u) = \sum_{i=1}^N w_{iN}(y(t_i) - f(t_i))^2 + \lambda L^N(u).$$

Now

$$J^N(u_0^N) = \lambda L(u_0^N)$$

and we have that for every λ, because of the optimality of u_λ^N,

$$J^N(u_\lambda^N) = \sum_{i=1}^N w_{iN}(y(t_i) - f(t_i))^2 + \lambda L(u_\lambda^N) \leq \lambda L(u_0^N) \tag{4.2}$$

and, because

$$\lim_{\lambda\to 0} \lambda L^N(u_0^N) = 0,$$

we conclude that

$$\lim_{\lambda\to 0}\sum_{i=1}^{N} w_{iN}(y(t_i)-f(t_i))^2=0.$$

We also have from equation 4.2 that

$$0\le\sum_{i=1}^{N} w_{iN}(y(t_i)-f(t_i))^2\le\lambda(L^N(u_0^N)-L^N(u_\lambda^N))$$

and hence we have for every λ

$$L^N(u_0^N)\ge L^N(u_\lambda^N).$$

We conclude that there exists a $u^*\in L^2[0,T]$ such that

$$\lim_{\lambda\to 0} u_\lambda^N=u^*.$$

Thus we have that $L^N(u^*)\ge L^N(u_0^N)$ since u_0^N is optimal, but on the other hand we have

$$L^N(u_0^N)\ge L^N(u_\lambda^N)$$

for every λ and so $L^N(u_0^N)=L^N(u^*)$ and by uniqueness

$$u_0^N=u^*$$

In [13] it was shown that indeed the smoothing splines are smoothing under some fairly gentle conditions on the error. The theorem was proved under the assumptions of normality but it is fairly clear that these restrictions are unnecessary and that weaker conditions, in the form of symmetry of the distribution of error, would suffice. This result establishes that the solution of Problem 2 converges to the solution of Problem 3 as N becomes large.

THEOREM 4.2 ([13]). *Let the t_i's be such that the*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N} w_{iN}f(t_i)=\int_0^T g(t)dt$$

for every continuous function, g. Let the error ϵ_i be identically independently normally distributed. The optimal solution of

$$J_\epsilon^N=\frac{1}{N}\sum_{i=1}^{N} w_{iN}(y(t_i)-\epsilon_i)^2+\lambda\int_0^T u(t)^2dt$$

converges to the optimal solution of

$$J_\lambda(u)=\int_0^T (y(t)-f(t))^2+\lambda u(t)^2dt.$$

The first result that was obtained was that the solutions of Problem 3 converge to the solutions of Problem 4 under certain conditions on the eigenvalues of the state matrix. It is probably true that the restriction on the eigenvalues can be relaxed but it has not been shown.

THEOREM 4.3 ([16]). *Assume that the matrix A has only real eigenvalues and let $f(t)$ be a C^∞ function on an interval that contains $[0,T]$. Let the t_i's be such that the*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N} w_{iN} f(t_i) = \int_0^T g(t)dt$$

for every continuous function, g. Then the sequence of controls $\{u_N(t)\}_{m=1}^{\infty}$ converges to the function $u_\lambda(t)$, which is optimal solution of Problem 4, in the L_2 norm.

In the following three theorems when we say that a sequence converges to a solution of Problem 5 we mean this in the sense that if the solution of Problem 5 exists then we have convergence, but if the solution of Problem 5 does not exist then the convergence is in the sense that the cost function of Problem 5 goes to zero when evaluated at the points of the sequence.

In this paper we have proved the following result

THEOREM 4.4. *Under the assumptions of Theorem 3.1 the solution of Problem 4 converges to the solution of Problem 5 as λ goes to 0.*

The main conclusion of this paper is that the following theorem holds.

THEOREM 4.5. *Under the assumptions of Theorems 4.1, 4.2 and 4.3 the solutions of Problem 2, $u^N_{\epsilon,\lambda}$ converge in $L^2[0,T]$ norm to the solution of Problem 5, u^*.*

Thus we have that the generalized smoothing splines smooth the data and recover the underlying function in the limit. As a corollary we also have

THEOREM 4.6. *Under the assumptions of Theorems 4.1, 4.2 and 4.3 the solutions of Problem 1 converge to the solution of Problem 5 in $L^2[0,T]$.*

The diagram in Figure 4.1 shows the relationship between the various problems. All convergence is with respect to $L^2[0,T]$.

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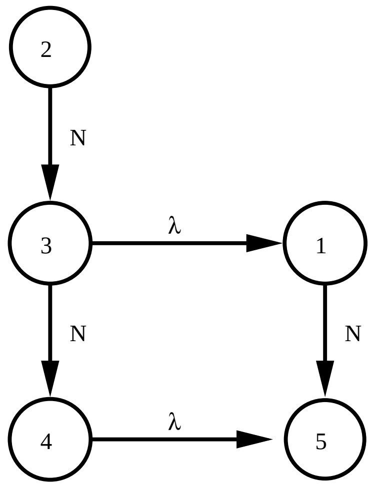

FIG. 4.1. *Relationships Between the Five Problems*

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