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POWER CONTROL UNDER FINITE POWER CONSTRAINTS*

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1. Introduction. In wireless communication, it is common that multiple users share the same radio spectrum simultaneously. Signal of one channel appears then as noise to the other channels. Hence, users transmitting power at power levels above than what is necessary to maintain a given level of signal-to-noise quality are not only wasteful of their battery power, they also cause unnecessary noise to other users. As a result, other users may try to compensate their signal quality by increasing their transmission power. This racing condition due to mutual interference could lead to an unstable mode in which all users try to raise their power levels without achieving any real gain in the overall signal-to-noise ratio. The objective of the *power control problem* is to ameliorate these adverse mutual interference effects (see for example [1-3]) as well as effects from fading effects and thermal noises.

Generally speaking, the power control problem should be viewed as a distributed, nonlinear, constrained optimal control problem. In certain cases, it can also be regarded as a multi-user game-theoretic problem. Due to the problem complexity, it is common to simplify it to make the problem amendable to analysis. For a classification of the power control problems usually discussed in the literature, we refer to [4].

A crucial structure of the power control problem is the mapping between the power levels and the carrier-to-interfere ratio (essentially, the signal-to-noise ratio). Let G_{ij} represent the channel gain between the i-th receiver and the j-th transmitter. In practice, this gain is an unknown time-varying parameter, depending on the distance between the transmitter and receiver as well as on the fading effects. The matrix $G = (G_{ij})$ is known as the channel gain matrix. The carrier-to-interference ratio of the i-th receiver is defined as

$$\Gamma_i(P) = \frac{G_{ij}P_i}{\sum\limits_{j\neq i} G_{ij}P_j + \tau_i}, i = 1, 2, \ldots m \tag{1}$$

where $P = (P_1, \ldots P_m)$ denotes the power vector and $\tau = (\tau_1, \ldots \tau_m)$ denotes the thermal noise vector.

A common formulation of the power control problem is to consider it as a max-min problem

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$$\min_{1\leq i\leq m}\left(\Gamma_1\left(P\right)\ ,\cdots\ \Gamma_m\left(P\right)\right)\rightarrow \max,$$

subject to obvious constraints $P_i \geq 0$, $i = 1, 2, \ldots m$. The gain matrix G is unknown but assumed to be a constant.

In this paper, we consider a model with additional physical constraints on the power vector P. As a key example of such constraints, we consider $P_i \leq M_i < \infty$, $i = 1, 2, \ldots m$. In other words, we assume that the transmission powers of all channels are bounded. This constraint makes the model more realistic. Moreover, the solution to an unconstrained formulation with non-zero noise terms is not well defined.

We describe the structure of the optimal solution of the arising optimization problem, provide a sensitivity analysis of this solution and outline a decentralized algorithm for the actual calculation of this solution.

2. Structure of the optimal solution and a sensitivity analysis. Assuming that diagonal entries G_{ii} of the gain matrix G are positive, we can introduce the normalized link gain matrix $Z = (Z_{ij})$, $Z_{ij} = G_{ij}/G_{ii}$. The ratio coefficients Γ_i will take the form

$$\Gamma_i\left(P\right)=\frac{P_i}{\sum\limits_{j\neq i} Z_{ij}P_j+\tau_i}, i=1,2,\ldots m. \tag{2}$$

Our basic optimization problem has the form

$$\min_{1\leq i\leq m}\left(\Gamma_1\left(P\right)\ ,\cdots\ \Gamma_m\left(P\right)\right)\rightarrow \max, \tag{3}$$

$$0\leq P_i\leq M_i, i=1,2,\ldots m. \tag{4}$$

We start by describing an optimal solution of the problem defined by (3) and (4) for a very special case.

PROPOSITION 1. *Let $M_i > 0, \tau_i > 0, i = 1, 2, \ldots m, Z = I$. Suppose (without loss of generality) that*

$$\frac{M_1}{\tau_1}\leq\frac{M_2}{\tau_2}\leq\cdots\leq\frac{M_m}{\tau_m}. \tag{5}$$

Then

$$P_1^*=M_1,\quad P_2^*=\frac{\tau_2}{\tau_1}M_1,\quad \ldots\quad P_m^*=\frac{\tau_m}{\tau_1}M_1$$

is an optimal solution to the max-min problem defined by (3) and (4) such that

(6) $$\Gamma_1 (P^*) = \Gamma_2 (P^*) = \cdots = \Gamma_m (P^*) .$$

Proof. Due to (5) it is obvious that $P_i^* \leq M_i$, $i = 1, 2, \ldots m$. Since $Z = 0$,

$$\Gamma_i (P) = \frac{P_i}{\tau_i} \leq \frac{M_i}{\tau_i}, i = 1, 2, \ldots m,$$

for any P satisfying (4). Hence,

$$\min_{i=1,2,\cdots m} \Gamma_i (P) \leq \min_{i=1,2,\cdots m} \frac{M_i}{\tau_i} = \frac{M_1}{\tau_1}.$$

But

$$\Gamma_1 (P^*) = \Gamma_2 (P^*) = \cdots = \Gamma_m (P^*) = \frac{M_1}{\tau_1}.$$

Hence, the result follows. □

Observe that in general, i.e. without the assumption $Z = 0$, we have:

(6a) $$f(P) = \min_i \Gamma_i (P) = \min_i \frac{P_i}{\sum\limits_{j \neq i} Z_{ij} p_j + \tau_i} \leq \min_i \frac{M_i}{\tau_i} = \frac{M_1}{\tau_1}.$$

Hence, we obtain the first upper bound for an optimal value for the problem defined by (3) and (4). This result is summarized below.

PROPOSITION 2. *Let P satisfy constraints (4) and $Z_{ij} \geq 0$, i, $j = 1, 2, \ldots m$. Then*

(7) $$\min_i \Gamma_i (P) \leq \frac{M_1}{\tau_1}.$$

The next proposition describes an important qualitative property of an optimal solution to (3) and (4).

PROPOSITION 3. *Suppose that $\tau_i > 0$, $M_i > 0$, $Z_{ij} \geq 0$, i, $j = 1, 2, \ldots m$. Let P^* be an optimal solution to (3) and (4). Then $P_i^* = M_i$ for at least one i.*

Proof. It is obvious that the maximal value for (3) and (4) is positive. Hence, $P_i^* > 0$, $\forall i = 1, 2, \ldots m$. Suppose that $P_i^* < M_i$, $\forall i$.

Choose $t > 0$ such that $tP_i^* < M_i$, $\forall i$. We have

$$\Gamma_i (tP^*) = \frac{P_i^*}{\sum\limits_{j \neq i} Z_{ij} p_j^* + \tau_i / t}.$$

Hence, $\Gamma_i (tP^*) > \Gamma_i (P^*)$, $\forall i$. This obviously implies that

$$f(tP^*) > f(P^*).$$

A contradiction. Hence, $P_i^* = M_i$ for at least one i. □

The next proposition extends a classical result on the optimal solution to our model. It offers an important characterization of the optimal solution to (3) and (4).

Proposition 4. *Let $M_i > 0$, $\tau_i > 0$, $Z_{ij} > 0$, $i, j = 1, 2, \ldots m$. Then for any optimal solution P^* to (3) and (4) we have:*

$$\Gamma_1(P^*) = \Gamma_2(P^*) = \cdots = \Gamma_m(P^*) = \nu. \tag{8}$$

Conversely, if P^* is a solution to

$$\Gamma_1(P) = \Gamma_2(P) = \cdots = \Gamma_m(P) = c \tag{9}$$

for some c, with $0 \leq P_j^* \leq M_j$ for all j and $P_i^* = M_i$ for at least one i, then P^* is an optimal solution to the max-min problem. That is:

$$c = \max_{\substack{0 \leq P_i \leq M_i \\ \forall i}} f(P)$$

Moreover, the optimal solution is unique.

Proof. Suppose that, say, $\Gamma_1(P^*) \neq \nu = f(P^*)$. If $\Gamma_1(P^*) < \nu$, then

$$f(P^*) = \min_i \Gamma_i(P^*) \leq \Gamma_i(P^*) < \nu = f(P^*).$$

A contradiction. Suppose that $\Gamma_1(P^*) > \nu$. Obviously $\nu > 0$ and hence $P^* > 0$, $\forall i$. Consider the functions

$$\Delta_i(\varepsilon) = \Gamma_i(P_1^* - \varepsilon,\ P_2^*,\ \cdots\ P_m^*), i = 1, 2, \ldots m, \varepsilon \geq 0.$$

By (2) and our assumptions:

$$\Delta_1(\varepsilon) < \Gamma_1(P^*), \quad \Delta_i(\varepsilon) \geq \Gamma_i(P^*), i = 1, 2, \ldots m,$$

for sufficiently small $\varepsilon > 0$. Since $\Delta_1(0) = \Gamma_1(P^*) > \nu$, we obviously have $\Gamma_1(P^*) > \Gamma_i(P^*)$ for any i such that $\Gamma_i(P^*) = \nu$. Then we can choose $\varepsilon > 0$ so small that $\Delta_1(\varepsilon) > \Delta_i(\varepsilon)$ for any i such that $\Gamma_i(P^*) = \nu$. Then

$$\min_{1 \leq i \leq m} \Delta_i(\varepsilon) = \min_{2 \leq i \leq m} \Delta_i(\varepsilon) > \min_{2 \leq i \leq m} \Delta_i(0) = \nu,$$

since functions $\Delta_i(\varepsilon)$ are monotonically increasing for $i \geq 2$. Here we used the assumption that $Z_{ij} > 0$ for all i, j.
Thus

$$f(P_1^* - \varepsilon,\ P_2^*,\ \cdots\ P_m^*) > f(P^*).$$

Contradicts the fact that $f(P^*)$ is the maximal value of on the feasible set (4).

To prove the converse, first note that conditions (9) can be rewritten in following matrix form:

$$P = c((Z - I)P + \tau) \tag{10}$$

where $P = (P_1,\ \cdots\ P_m)^T$, $\tau = (\tau_1,\ \tau_2\ \cdots\ \tau_m)^T$, and I is the identity $m \times m$ matrix. Denote $Z - I$ by $\tilde{Z}$. We can rewrite (10) in the form:

$$\left(\frac{1}{c}I - \tilde{Z}\right)P = \tau \tag{11}$$

The solution can be expressed in the form (see for example [5]):

$$P = \left(\frac{1}{c}I - \tilde{Z}\right)^{-1}\tau = c\sum_{k=0}^{\infty}(c\tilde{Z})^k\tau. \tag{12}$$

If the optimal solution is larger than c, that is, $\nu > c$, then the corresponding power level vector, P', satisfies $P' > P$. But then P'_i violates the power constraint. A contradiction. The uniqueness of the solution also follows from (12). $\square$

REMARK. *The result holds true (along with the proof) if we add any number of constraints of the form:*

$$\sum_{i=1}^{m} a_i P_i \leq C,$$

where $a_i \geq 0$, $i = 1,\ 2,\ \ldots m$, provided the resulted feasible set has a nonempty interior. An example of such a constraint may be that the total received power should be bounded.

REMARK. *(9) appears to consist of m equations with $m+1$ unknowns. However, by Proposition 3, if P is an optimal solution, then $P_i = M_i$ for at least one i. Hence, one way to find the optimal solution is to consider m sets of m equations in m unknowns by setting $P_i = M_i$ for i equals to 1, 2, ... and so on.*

Let $\lambda\left(\tilde{Z}\right)$ be the spectral radius of $\tilde{Z}$. By Perron-Frobenius theorem $\lambda\left(\tilde{Z}\right)$ is a positive eigenvalue of $\tilde{Z}$. There exists right q_1 and left q_2 eigenvectors of $\tilde{Z}$ with positive coordinates ([5]).

The next proposition provides the second upper bound for the maximal value of (3), (4).

PROPOSITION 5. *Let P satisfy constraints (4) and $Z_{ij} > 0\ \forall i, j$. Then*

$$\min \Gamma_i(P) < \frac{1}{\lambda\left(\tilde{Z}\right)}.$$

Proof. The result easily follows from the Perron-Frobenius theorem. Let us sketch the proof .Let P^* be an optimal solution of (3), (4) and ν^* be the optimal value of the cost function (3). Then by (11)

$$\left(I - \nu^* \tilde{Z}\right) P^* = \nu^* \tau.$$

Hence

$$q_1^T \left(I - \nu^* \tilde{Z}\right) P^* = \nu^* q_1^T \tau, \tag{13}$$

where q_1 is the right eigenvector of $\tilde{Z}$ with positive coordinates corresponding to the eigenvalue $\lambda\left(\tilde{Z}\right)$.

Since $\nu^* > 0$ and all coordinates of P^* are positive, we derive from (12):

$$q_1^T P^* \left(1 - \nu^* \lambda\left(\tilde{Z}\right)\right) > 0.$$

Hence,$1 - \nu^* \lambda\left(\tilde{Z}\right) > 0$. The result follows. □

3. Some perturbation results. Our next proposition shows that the optimal solution described in Proposition 1 is robust under small feasible perturbations of the matrix $\tilde{Z}$.

Given $x \in \Re^m$, $x = (x_1, \cdots x_m)^T$, let

$$\|x\|_\infty = \max_{1 \le i \le m} |x_i|.$$

If A is an $m \times m$ matrix, we denote by $\|A\|_\infty$ the corresponding matrix norm.

PROPOSITION 6. *Let*

$$\frac{M_1}{\tau_1} \le \frac{M_2}{\tau_2} \le \cdots \le \frac{M_m}{\tau_m}.$$

Assume $\left\|\tilde{Z}\right\|_\infty$ is so small that

$$\frac{1}{\tau_i}\left(\frac{M_1}{\tau_1}\right)^2 \|\tau\|_\infty \left\|\tilde{Z}\right\|_\infty \left\|\left(I - \nu^* \tilde{Z}\right)^{-1}\right\|_\infty < \frac{M_i}{\tau_i} - \frac{M_1}{\tau_1}, \quad \forall i = 2, 3, \ldots, m.$$

If $P_1^*, \cdots P_m^*$ *is an optimal solution to (3) and (4), then*

$$P_1^* = M_1 \quad and \quad P_i^* < M_i, i = 2, 3, \dots m.$$

Proof. Let $\nu^* = \min_i \Gamma_i (P^*)$. Using (11) and Proposition 5, we have

$$P^* = \nu^* \left(I + \nu^* \tilde{Z} + (\nu^*)^2 \tilde{Z}^2 + \cdots \right) \tau, \tag{14}$$

where the series in the right-hand side of (14) converges absolutely. This can be rewritten in the form:

$$P^* - \nu^* \eta = (\nu^*)^2 \tilde{Z} \left(I - \nu^* \tilde{Z} \right)^{-1} \tau.$$

Hence,

$$\begin{aligned} \| P^* - \nu^* \tau \|_\infty &\le (\nu^*)^2 \left\| \tilde{Z} \right\|_\infty \left\| \left(I - \nu^* \tilde{Z} \right)^{-1} \right\|_\infty \| \tau \|_\infty \\ &\le \left(\frac{M_1}{\tau_1} \right)^2 \left\| I - \nu^* \tilde{Z} \right\|_\infty \| \tau \|_\infty = \Delta. \end{aligned}$$

Here we used (7). In particular,

$$P_i^* - \nu^* \tau_i \le \Delta, i = 2, 3, \dots m.$$

or

$$P_i^* \le \nu^* \tau_i + \Delta \le \frac{M_1}{\tau_1} \tau_i + \Delta,$$

which implies

$$\frac{P_i^*}{\tau_i} \le \frac{M_1}{\tau_1} + \frac{\Delta}{\tau_i} < \frac{M_i}{\tau_i}, i = 2, 3, \dots m.$$

Using Proposition 3, we conclude that

$$P_1^* = M_1.$$

□

We now consider another limiting situation, which corresponds to the case of a small noise. Let $\tau = 0$. Then the optimality condition has the form (see (11)):

$$P^* = \nu^* \tilde{Z} P^*,$$

i.e. P^* is a positive eigenvector corresponding to the eigenvalue $\lambda \left(\tilde{Z} \right)$ and $\nu^* = \frac{1}{\lambda(\tilde{Z})}$. Suppose that $P_s^* = M_s$ for some $s \in [1, m]$ and $P_i^* < M_i$, $i \ne s$.

Proposition 7. *For sufficiently small $\tau > 0$, we have for the optimal solution $P^* (\tau)$:*

$$P_s^* (\tau) = M_s, \quad P_i^* (\tau) < M_i, \quad i \neq s.$$

Proof. By (11), we have:

$$P^* (\tau) = \nu^* (\tau) \tilde{Z} P^* (\tau) + \nu^* (\tau) \tau.$$

Since $P^* = P(0)$ is a simple eigenvector of $\tilde{Z}$ (by Perron-Frobenius Theorem, see e.g. [5]) and $P^* (\tau)$ depends analytically on τ for sufficiently small τ. In particular, $P_i^* (\tau) = M_i$, $i \neq s$ for sufficiently small τ. Hence, $P_s^* (\tau) = M_s$ by Proposition 3. □

4. Decentralized algorithm based on dichotomy. Suppose that we know the optimal value ν^* for the problem defined by (3) and (4). Then to find the optimal solution we need to solve (11). A well-known iterative scheme is:

$$P^{(n+1)} = \left(\nu^* \tilde{Z}\right) P^{(n)} + \nu^* \tau. \tag{15}$$

This scheme converges to the solution of (11) since $\nu^* < \frac{1}{\lambda(\tilde{Z})}$ by Proposition 5. We can rewrite (15) in the form:

$$P_i^{(n+1)} = \nu^* \sum_{j \neq i} z_{ij} P_j^{(n)} + \nu^* \tau_i \tag{16}$$

or

$$P_i^{(n+1)} = \frac{\nu^* P_i^{(n)}}{\Gamma_i \left(P^{(n)}\right)}, i = 1, 2, \ldots m. \tag{17}$$

Now consider a power control problem for uplink channels. That is, assume m users are sending information to the base stations. Assume that the power levels may temporarily exceed the constraints M_i's. However, we would like to find stationary solutions that satisfy the constraints. A viable algorithm is to require users adjust their power levels by the algorithm defined in (17). The base stations then estimate the carrier-to-interference ratios and communicate them to the corresponding mobile users for the next computation iteration. This provides a decentralized (or distributed) version for the finding the optimal solution to (3) and (4). The only problem is that we do not know ν. However, there is a simple variation of this approach that requires only minimal interaction among the base stations.

Assume that a stopping rule can be defined to detect whether the iterative algorithm defined in (17) is converging or not. For example, the base station can monitor whether the carrier-to-interference ratios have been unchanging for a given period.

Set $\underline{\nu} = 0$ (or any lower bound for ν^*) and take $\overline{\nu}$ to be equal to a known upper bound of ν^* (see Proposition 2, 5).

Take $\nu = \frac{\underline{\nu}+\overline{\nu}}{2}$. ν is broadcast to all users and each users will update their power according to (17) with ν^* replaced by ν. Notice that the updates can be done completely in a distributed manner, once ν is given. Let $P = (P_1, \cdots P_m)$ be the solution to (17) (observe that $P_i \geq 0$, $i = 1, 2, \ldots m$). If a) $P_i > M_i$ for at least one i, set $\underline{\nu} = \underline{\nu}$, $\overline{\nu} = \nu$, and repeat the procedure. If b) $P_i < M_i$ for all i, set $\underline{\nu} = \nu$, $\overline{\nu} = \overline{\nu}$ and repeat the procedure. If $P_i \leq M_i$ for all i, with at least one equality, then stop. In this case $(P_1, \cdots P_m)$ is an optimal solution and the corresponding ν is an optimal value for (3) and (4).

The justification for this procedure is very simple. In case a) we can guarantee that $\nu^* \in [\underline{\nu}, \nu)$ and in case b) we can guarantee that $\nu^* \in (\nu, \overline{\nu}]$.

Initially, we know that $\underline{\nu} \leq \nu^* \leq \overline{\nu}$. In case of a), we know that $\underline{\nu} \leq \nu^* < \frac{\overline{\nu}+\underline{\nu}}{2}$. Since if $\nu^* \geq \frac{\overline{\nu}+\underline{\nu}}{2}$, then the corresponding optimal solution $P^* \geq P$ as it follows from (12) and hence, cannot be feasible.

In case of b) $\frac{\overline{\nu}+\underline{\nu}}{2} < \nu^* \leq \overline{\nu}$. Indeed, if $\nu^* < \frac{\underline{\nu}+\overline{\nu}}{2}$, then P would be a feasible solution with the value of cost function "better than optimal". The case $\nu^* = \frac{\underline{\nu}+\overline{\nu}}{2}$ is impossible by Proposition 3. We can proceed by induction to show that $\underline{\nu} \leq \nu^* \leq \overline{\nu}$ on each iteration.

Since $\underline{\nu} - \overline{\nu}$ converges to zero very rapidly, our dichotomy procedure should work fine.

Remark. *The proposed algorithm works with any number of additional constraints of the form:*

$$\sum_{i=1}^{m} a_i P_i \leq C, \text{ where } \; a_i \geq 0,$$

provided that the feasible domain has a nonempty interior.

5. Numerical Results. To test the effectiveness of the dichotomy approach, the following numerical simulation is carried out. There are 5 users in 5 cells. The maximum power restriction is given by: [1, 1.4 , 2.0 , 1.8 , 1.6]. The thermal vector is given by: [0.1 , 0.1 , 0.11 , 0.2 , 0.07]. The channel gain values are generated randomly according. To initialize the system the power of all users are zero at the beginning. Moreover,

$$v = 0 \quad \bar{v} = \min_i \frac{M_i}{\tau_i}$$

When $|P_i - M_i| < 10^{-8}$ for all i, we consider that the system converges.

In the following figures, one can see that the algorithm converges rather quickly.

6. Concluding remark. In the present paper we have considered the power control problem with additional constraints on the power vector. Surprisingly, the structure of the optimal solution retains major features of the unconstrained problem.

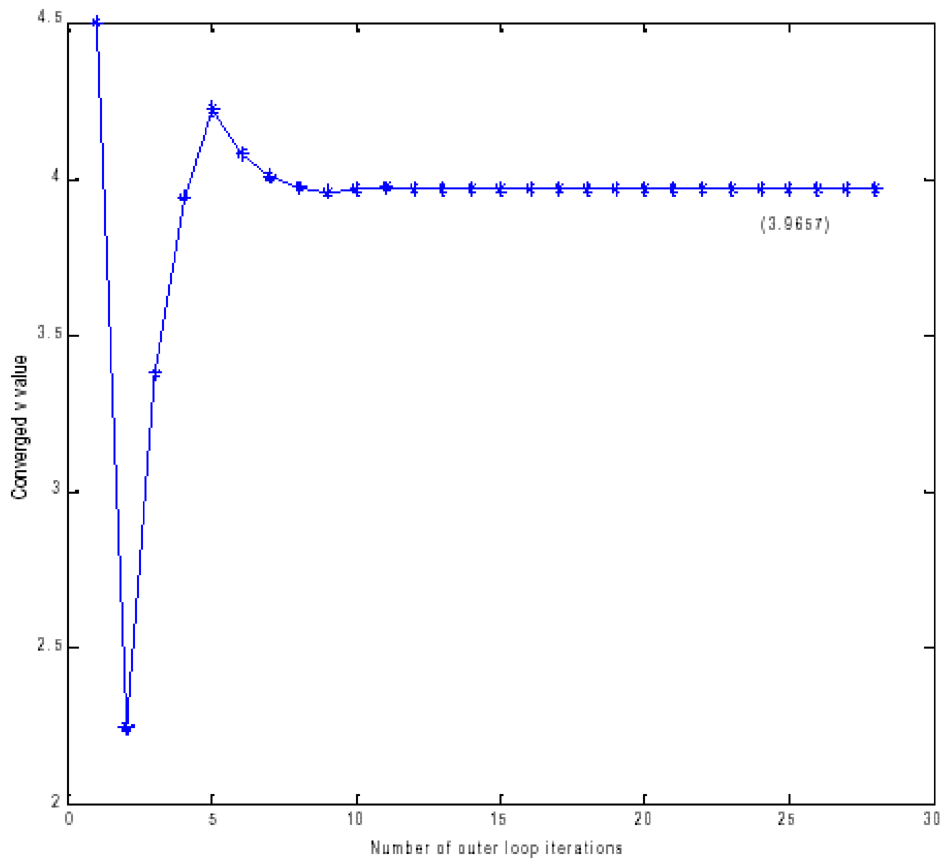

Fig. 1. *Convergence of the v value*

We have provided a complete analysis of the structure of the optimal solution, its partial sensitivity analysis and described a distributed algorithm based on a dichotomy principle. I would like to thank Ge Wei Yan for the simulation work for this paper. Numerous discussions with Wing Shing Wong are greatly appreciated. When this paper has been completed, the following reference has been brought to the author's attention: "Constrained power control in cellular radio systems" by Grandhi, S.A. and J. Zander in Vehicular Technology Conference, 1994 IEEE 44th, 1994 Page(s): 824 -828 vol.2, where a constrained power control problem is also considered. However, our model is more general (we allow different constraint levels for different users), we provide a stability analysis of an optimal solution and our decentralized algorithm is new and provides a fast convergence to the optimal solution. Observe that our results (including the algorithm) are applicable to much more general constraints.

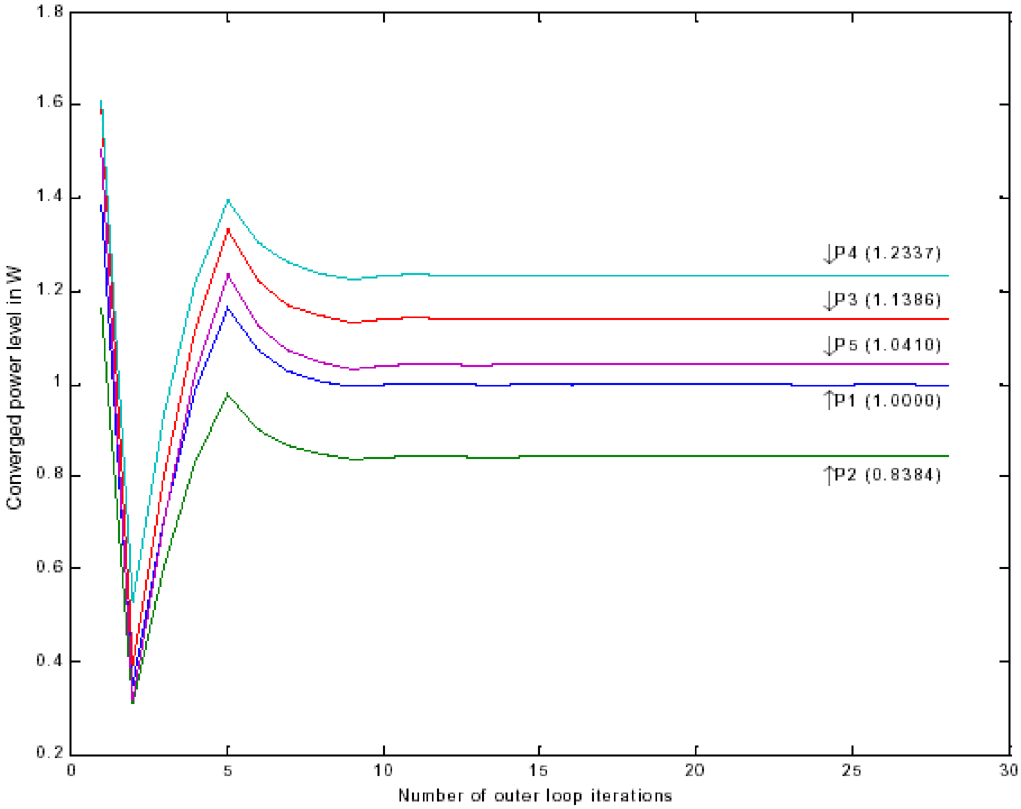

FIG. 2. *Convergence of the Power Levels*

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