SMALL GAIN THEOREM WITH RESTRICTIONS FOR UNCERTAIN TIME-VARYING NONLINEAR SYSTEMS

MINGHUI ZHU* AND JIE HUANG†

Abstract. This paper gives three versions of the small gain theorem with restrictions for uncertain time-varying nonlinear systems. The result can be viewed as an extension of the small gain theorem with restrictions for time-invariant nonlinear systems or the small gain theorem without restrictions for time-varying nonlinear systems. The result can be applied to study the stabilization problem or the output regulation problem of uncertain nonlinear systems.

Index Terms – small gain theorem, nonlinear control, nonlinear systems.

The small gain theorem is an important tool to ascertain the stability of two interconnected systems assuming each of the individual systems is stable in some sense. Small gain theorem has many different versions under various stability concepts [2] to [14], [20]. In this paper, we will focus on the small gain theorem in the context of input-to-state and/or input-to-output stability [15] to [19]. The first small gain theorem for nonlinear time-varying systems in the input-to-state stability (ISS) framework was established by Jiang *et al* [7]. The resulting small gain condition is given in terms of two inequalities. Recently, Chen and Huang further considered the small gain theorem for uncertain time-varying nonlinear system [2]. They presented a simplified small gain condition which is in a familiar form of the contraction mapping known for time-invariant nonlinear systems [7].

In [20], Teel introduced the concept of ISS with restrictions on the initial states and inputs and established a small gain theorem with restrictions for time-invariant systems. In Appendix B of [6], relying upon the separation property for ISS with restrictions, Isidori *et al* established a more general small gain theorem with restrictions for time-invariant systems. Nevertheless, the proof of [6] cannot be carried over to the case of time-varying systems, because the separation property for ISS does not hold for time-varying systems [2].

This paper is to establish three versions of the small gain theorem with restrictions for uncertain time-varying nonlinear systems, thus filling the gap between the small gain theorem with restrictions for time-varying nonlinear systems and that for time invariant nonlinear systems.

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1. Preliminary. Throughout the paper, let L_∞^m be the set of all piecewise continuous bounded functions $u : [t_0, \infty) \mapsto \Re^m$ with a finite supremum norm $\|u_{[t_0,\infty)}\| = \sup_{t \geq t_0} \|u(t)\|$. Denote the supremum norm of the truncation of $u(t)$ in $[t_1, t_2]$ by $\|u_{[t_1,t_2]}\| = \sup_{t_1 \leq t \leq t_2} \|u(t)\|$. And denote $\|u\|_a = \limsup_{t \to \infty} \|u\|$. A continuous function $\gamma : \Re_{\geq 0} \mapsto \Re_{\geq 0}$ is of class K if it is strictly increasing and $\gamma(0) = 0$; and a continuous function $\beta(s, t) : \Re_{\geq 0} \times \Re_{\geq 0} \mapsto \Re_{\geq 0}$ is of class KL if, for each fixed $t \geq 0$, the function $\beta(s, t)$ belongs to class K with respect to s and, for each fixed s, the function $\beta(s, t)$ is decreasing with respect to t, and $\beta(s, t) \to 0$ as $t \to \infty$.

Consider the following time-varying uncertain nonlinear system

$$\dot{x} = f(x, u, d, t),$$

(1)
$$y = h(x, u, d, t) \quad t \geq t_0 \geq 0$$

where $x \in \Re^n$ is the plant state, $u \in \Re^m$ the input, $y \in \Re^p$ the output, t_0 the initial time, the functions $f : \Re^n \times \Re^m \times \Re^{n_d} \times [t_0, \infty) \mapsto \Re^n$ and $h : \Re^n \times \Re^m \times \Re^{n_d} \times [t_0, \infty) \mapsto \Re^p$ are piecewise continuous in t and locally *Lipschitz* in $\mathrm{col}(x, u, d)$. And $d(t) : [t_0, \infty) \mapsto \Re^{n_d}$ is a family of piecewise continuous functions of t, representing the external disturbance and/or the internal uncertainty.

DEFINITION 1.1.   System (1) is said to be (uniformly) robust input-to-state stable (RISS) with restrictions $X \subset \Re^n$ and $\Delta > 0$ on the initial state $x(t_0)$ and the input u respectively if there exist class KL function β and class K function γ, independent of $d(t)$, such that, for any initial state $x(t_0) \in X$ and any input function $u(t) \in L_\infty^m$ satisfying $\|u_{[t_0,\infty)}\| < \Delta$, the solution of (1) exists and satisfies, for all $t \geq t_0$,

$$\|x(t)\| \leq \max\{\beta(\|x(t_0)\|, t - t_0), \gamma(\|u_{[t_0,t]}\|)\}.$$

□

DEFINITION 1.2.   System (1) is said to be robust input-to-output stable (RIOS) with restrictions X and Δ on the initial state $x(t_0)$ and the input u respectively if there exist class KL function β and class K function γ, independent of $d(t)$, such that, for any initial state $x(t_0) \in X$, any input function $u(t) \in L_\infty^m$ satisfying $\|u_{[t_0,\infty)}\| < \Delta$, the output of (1) exists and satisfies, for all $t \geq t_0$,

$$\|y(t)\| \leq \max\{\beta(\|x(t_0)\|, t - t_0), \gamma(\|u_{[t_0,t]}\|)\}.$$

□

DEFINITION 1.3.   System (1) is said to be semi-uniformly RISS with restrictions X and Δ on the initial state $x(t_0)$ and the input u respectively if there exist class K functions γ^0 and γ^u, independent of $d(t)$, such that for any initial state $x(t_0) \in X$

and input $u \in L_\infty^m$ satisfying $\|u_{[t_0,\infty)}\| < \Delta$, the solution of (1) exists and satisfies, for all $t \geq t_0$,

$$\|x(t)\| \leq \max\{\gamma^0(\|x(t_0)\|), \gamma^u(\|u_{[t_0,\infty)}\|)\}$$

(2) $$\|x\|_a \leq \gamma^u(\|u\|_a).$$

$\square$

REMARK 1.1.   In [17], it was shown that, for the class of time-invariant systems, ISS is equivalent to semi-uniformly ISS. Such equivalence is called *separation property*. This equivalent relation can also be extended to ISS with restrictions and semi-uniformly ISS with restrictions (Appendix B of [6]). Unfortunately, the *separation property* does not hold for the time-varying nonlinear systems [2]. $\square$

DEFINITION 1.4.   System (1) is said to satisfy robust asymptotic gain (RAG) property with restrictions X and Δ on the initial state $x(t_0)$ and the input u respectively if there exists a class K function γ^u, independent of $d(t)$, such that for any initial state $x(t_0) \in X$ and input $u \in L_\infty^m$ satisfying $\|u\|_a \leq \Delta$, the solution of (1) exists and satisfies, for all $t \geq t_0$,

(3) $$\|x\|_a \leq \gamma^u(\|u\|_a).$$

$\square$

DEFINITION 1.5.   The output function of (1) is said to satisfy robust asymptotic L_∞ stability (RALS) with restrictions X and Δ on the initial state $x(t_0)$ and the input u respectively if there exist class K functions γ^0 and γ^u, independent of $d(t)$, such that for any initial state $x(t_0) \in X$ and input $u \in L_\infty^m$ satisfying $\|u_{[t_0,\infty)}\| < \Delta$, the output of (1) exists and satisfies, for all $t \geq t_0$,

$$\|y(t)\| \leq \max\{\gamma^0(\|x(t_0)\|), \gamma^u(\|u_{[t_0,\infty)}\|)\}$$

(4) $$\|y\|_a \leq \gamma^u(\|u\|_a).$$

$\square$

DEFINITION 1.6.   System(1) is said to satisfy output robust asymptotic gain (o-RAG) property with restrictions X and Δ on the initial state $x(t_0)$ and the input u respectively if there exists class K function γ^u, independent of $d(t)$, such that for any initial state $x(t_0) \in X$ and input $u \in L_\infty^m$ satisfying $\|u\|_a \leq \Delta$, the output of (1) exists and satisfies, for all $t \geq t_0$,

(5) $$\|y\|_a \leq \gamma^u(\|u\|_a).$$

$\square$

2. Small Gain Theorem with Restrictions for Uncertain Nonlinear Time-varying Systems.

2.1. The Case of Time Invariant Nonlinear Systems. Consider the feedback interconnection as depicted in Figure 1,

(6) $$\dot{x}_1 = f_1(x_1, v_1, u_1), \quad y_1 = h_1(x_1, v_1, u_1)$$

(7) $$\dot{x}_2 = f_2(x_2, v_2, u_2), \quad y_2 = h_2(x_2, v_2, u_2)$$

subject to the following interconnection:

(8) $$v_1 = y_2, \quad v_2 = y_1$$

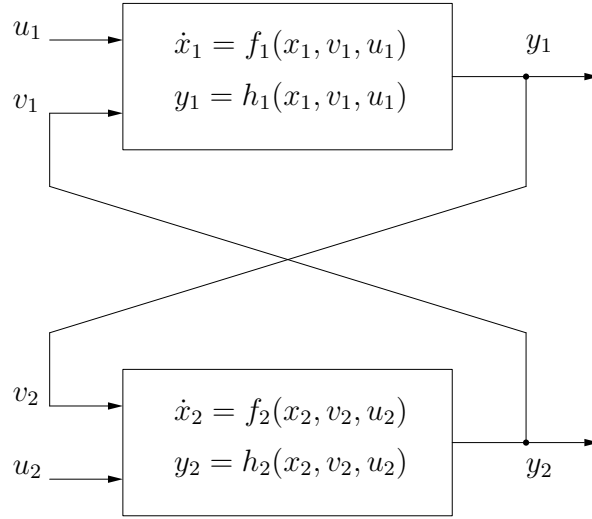FIG. 1. *Inter-connection of (6) and (7)*

where, for $i = 1, 2$, $x_i \in \Re^{n_i}$, $u_i \in \Re^{m_i}$, $y_i \in \Re^{p_i}$, $v_i \in \Re^{q_i}$ with $p_1 = q_2$, $p_2 = q_1$, the function $f_i(x_i, v_i, u_i)$ is locally *Lipschitz* in $\mathrm{col}(x_i, v_i, u_i)$, and $f_i(0, 0, 0) = 0$, $h_i(0, 0, 0) = 0$.

And suppose the following assumption holds.

Assumption 2.1. There exists a C^1 function h such that

$$\mathrm{col}\,(y_1, y_2) = h(x_1, x_2, u_1, u_2)$$

is the unique solution of the equations

$$y_1 = h_1(x_1, y_2, u_1), \quad y_2 = h_2(x_2, y_1, u_2).$$

The following small gain theorem with restrictions for time invariant nonlinear systems was established in Appendix B of [6].

THEOREM 2.1. Assume that subsystem (6) is ISS with restrictions X_1, Δ_1 and Δ_1^u on $x_1(0)$, v_1 and u_1 respectively and subsystem (7) is ISS with restrictions X_2, Δ_2 and Δ_2^u on $x_2(0)$, v_2 and u_2 respectively, i.e., there exist class KL functions β_1 and β_2, class K functions γ_1, γ_2, γ_1^u, γ_2^u such that, for any $x_1(0) \in X_1$, $v_1(t) \in L_\infty^{q_1}$ satisfying $\|v_{1[0,\infty)}\| < \Delta_1$, $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[0,\infty)}\| < \Delta_1^u$, the solution of (6) exists and satisfies, for all $t \geq 0$,

$$\|x_1(t)\| \leq \max\{\beta_1(\|x_1(0)\|, t), \gamma_1(\|v_{1[0,\infty)}\|), \gamma_1^u(\|u_{1[0,\infty)}\|)\}$$

and for any $x_2(0) \in X_2$, $v_2(t) \in L_\infty^{q_2}$ satisfying $\|v_{2[0,\infty)}\| < \Delta_2$, $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[0,\infty)}\| < \Delta_2^u$, the solution of (7) exists and satisfies, for all $t \geq 0$,

$$\|x_2(t)\| \leq \max\{\beta_2(\|x_2(0)\|, t), \gamma_2(\|v_{2[0,\infty)}\|), \gamma_2^u(\|u_{2[0,\infty)}\|)\}.$$

Suppose the following estimates hold for the outputs y_1 and y_2

$$\|y_{1[0,\infty)}\| \leq \max\{\bar{\gamma}_1^0(\|x_1(0)\|), \overline{\gamma}_1(\|v_{1[0,\infty)}\|), \overline{\gamma}_1^u(\|u_{1[0,\infty)}\|)\}$$
$$\|y_1\|_a \leq \max\{\overline{\gamma}_1(\|v_1\|_a), \overline{\gamma}_1^u(\|u_1\|_a)\}$$
$$\|y_{2[0,\infty)}\| \leq \max\{\bar{\gamma}_2^0(\|x_2(0)\|), \overline{\gamma}_2(\|v_{2[0,\infty)}\|), \overline{\gamma}_2^u(\|u_{2[0,\infty)}\|)\}$$
$$\|y_2\|_a \leq \max\{\overline{\gamma}_2(\|v_2\|_a), \overline{\gamma}_2^u(\|u_2\|_a)\}$$

for some class K functions $\bar{\gamma}_1^0$, $\bar{\gamma}_2^0$, $\overline{\gamma}_1$, $\overline{\gamma}_2$, $\overline{\gamma}_1^u$ and $\overline{\gamma}_2^u$.

Then if

$$\overline{\gamma}_1 \circ \overline{\gamma}_2(r) < r, \quad \forall r > 0$$

the system composed of (6) and (7) is ISS with restrictions $\tilde{X}_1 \times \tilde{X}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(0)$, u_1 and u_2 respectively, viewing $x = \mathrm{col}(x_1, x_2)$ as state and $u = \mathrm{col}(u_1, u_2)$ as input, i.e., there exist class KL function β and class K function γ, such that, for any initial state $x(0) \in \tilde{X}_1 \times \tilde{X}_2$, and any input functions $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[0,\infty)}\| < \tilde{\Delta}_1$ and $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[0,\infty)}\| < \tilde{\Delta}_2$, the solution of (6) and (7) under connection (8) exists and satisfies, for all $t \geq 0$,

(9) $$\|x(t)\| \leq \max\{\beta(\|x(0)\|, t), \gamma(\|u_{[0,\infty)}\|)\}$$

where,
$\tilde{X}_1 = \{x_1 \in X_1 : \bar{\gamma}_1^0(\|x_1\|) < \Delta_2, \overline{\gamma}_2 \circ \bar{\gamma}_1^0(\|x_1\|) < \Delta_1\}$
and

$\tilde{X}_2 = \{x_2 \in X_2 : \bar{\gamma}_2^0(\|x_2\|) < \Delta_1, \overline{\gamma}_1 \circ \bar{\gamma}_2^0(\|x_2\|) < \Delta_2\}$

$\tilde{\Delta}_1 \leq \Delta_1^u, \tilde{\Delta}_2 \leq \Delta_2^u$

$s \in [0, \tilde{\Delta}_1) \Longrightarrow \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s) < \Delta_1, \overline{\gamma}_1^u(s) < \Delta_2$

and

$s \in [0, \tilde{\Delta}_2) \Longrightarrow \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s) < \Delta_2, \overline{\gamma}_2^u(s) < \Delta_1.$ $\qquad\square$

REMARK 2.1.   Theorem 2.1 is slightly different from Theorem B.3.1 [6] where for $i = 1, 2$, $\gamma_i(s) = \overline{\gamma}_i(s)$ and $\gamma_i^u(s) = \overline{\gamma}_i^u(s)$. $\qquad\square$

2.2. The Case of Uncertain Time-varying Nonlinear Systems. Let us first introduce a technical lemma which was established in [2].

LEMMA 2.1.   Let β be a class KL function, γ a class K function such that $\gamma(r) < r$, $\forall r > 0$, and $\mu \in (0, 1]$ a real number. For any nonnegative real numbers s and M, and any nonnegative real function $z(t) \in L_\infty^1$ satisfying

$$z(t) \leq \max\{\beta(s, t), \gamma(\|z_{[\mu t, t]}\|), M\}, \quad \forall t \geq 0,$$

there exists a class K_∞ function $\hat{\beta}$ such that

$$z(t) \leq \max\{\hat{\beta}(s, t), M\}, \quad \forall t \geq 0.$$

$\qquad\square$

Consider the interconnection of the following two systems as depicted in Figure 2,

(10) $\qquad\qquad \dot{x}_1 = f_1(x_1, v_1, u_1, d, t), \quad y_1 = h_1(x_1, v_1, u_1, d, t)$

(11) $\qquad\qquad \dot{x}_2 = f_2(x_2, v_2, u_2, d, t), \quad y_2 = h_2(x_2, v_2, u_2, d, t)$

subject to the following interconnection:

(12) $\qquad\qquad\qquad\qquad v_1 = y_2, \quad v_2 = y_1$

where, for $i = 1, 2$, $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, $y_i \in \mathbb{R}^{p_i}$, $v_i \in \mathbb{R}^{q_i}$ with $p_1 = q_2$, $p_2 = q_1$, the functions $f_1(x_1, v_1, u_1, d, t)$ and $f_2(x_2, v_2, u_2, d, t)$ are piecewise continuous in t and locally *Lipschitz* in $\mathrm{col}(x_1, v_1, u_1, d)$ and $\mathrm{col}(x_2, v_2, u_2, d)$ respectively, and $d : [t_0, \infty) \mapsto \mathbb{R}^{n_d}$ is piecewise continuous.

The system composed of (10) and (11) is interpreted as feedback interconnection of two subsystems, the upper one with state x_1, input $\mathrm{col}(v_1, u_1)$ and output y_1 and the lower one with state x_2, input $\mathrm{col}(v_2, u_2)$ and output y_2. And suppose the following assumption holds.

Assumption 2.2. The equations

$$y_1 = h_1(x_1, h_2(x_2, y_1, u_2, d, t), u_1, d, t)$$

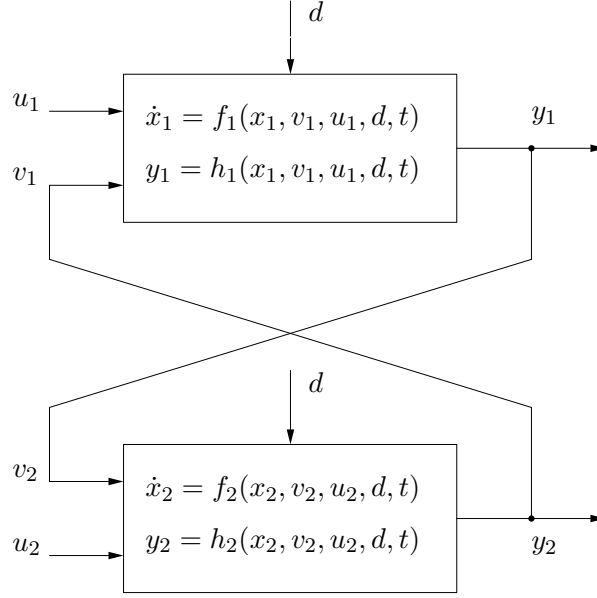$$y_2 = h_2(x_2, h_1(x_1, y_2, u_1, d, t), u_2, d, t)$$

FIG. 2. *Inter-connection of (10) and (11)*

have a unique solution of the form $y = h(x, u, d, t)$ where $x = \operatorname{col}(x_1, x_2)$, $y = \operatorname{col}(y_1, y_2)$, $u = \operatorname{col}(u_1, u_2)$, and h is locally *Lipschitz* in $\operatorname{col}(x, u, d)$ and piecewise continuous in t.

THEOREM 2.2. Assume that subsystem (10) is RISS with restrictions X_1, Δ_1 and Δ_1^u on $x_1(t_0)$, v_1 and u_1 respectively and subsystem (11) is RISS with restrictions X_2, Δ_2 and Δ_2^u on $x_2(t_0)$, v_2 and u_2 respectively, i.e., there exist class KL functions β_1 and β_2, class K functions γ_1, γ_1^u, γ_2, γ_2^u, independent of $d(t)$, such that, for any $x_1(t_0) \in X_1$, $v_1(t) \in L_\infty^{q_1}$ satisfying $\|v_{1[t_0,\infty)}\| < \Delta_1$, $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[t_0,\infty)}\| < \Delta_1^u$, the solution of (10) exists and satisfies, for all $t \geq t_0$,

$$(13) \qquad \|x_1(t)\| \leq \max\{\beta_1(\|x_1(t_0)\|, t - t_0), \gamma_1(\|v_{1[t_0,t]}\|), \gamma_1^u(\|u_{1[t_0,t]}\|)\}$$

and for any $x_2(t_0) \in X_2$, $v_2(t) \in L_\infty^{q_2}$ satisfying $\|v_{2[t_0,\infty)}\| < \Delta_2$, $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[t_0,\infty)}\| < \Delta_2^u$, the solution of (11) exists and satisfies, for all $t \geq t_0$,

$$(14) \qquad \|x_2(t)\| \leq \max\{\beta_2(\|x_2(t_0)\|, t - t_0), \gamma_2(\|v_{2[t_0,t]}\|), \gamma_2^u(\|u_{2[t_0,t]}\|)\}.$$

Further assume that subsystem (10) is RIOS with restrictions $\bar{X}_1$, $\overline{\Delta}_1$ and $\overline{\Delta}_1^u$ on $x_1(t_0)$, v_1 and u_1 respectively and subsystem (11) is RIOS with restrictions $\bar{X}_2$, $\overline{\Delta}_2$ and $\overline{\Delta}_2^u$ on $x_2(t_0)$, v_2 and u_2 respectively, i.e., there exist class KL functions $\overline{\beta}_1$ and $\overline{\beta}_2$,

class K functions $\overline{\gamma}_1, \overline{\gamma}_1^u, \overline{\gamma}_2, \overline{\gamma}_2^u$, independent of $d(t)$, such that, for any $x_1(t_0) \in \bar{X}_1$, $v_1(t) \in L_\infty^{q_1}$ satisfying $\|v_{1[t_0,\infty)}\| < \bar{\Delta}_1$, $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[t_0,\infty)}\| < \bar{\Delta}_1^u$, the output of (10) exists and satisfies, for all $t \geq t_0$,

$$(15) \qquad \|y_1(t)\| \leq \max\{\overline{\beta}_1(\|x_1(t_0)\|, t - t_0), \overline{\gamma}_1(\|v_{1[t_0,t]}\|), \overline{\gamma}_1^u(\|u_{1[t_0,t]}\|)\}$$

and for any $x_2(t_0) \in \bar{X}_2$, $v_2(t) \in L_\infty^{q_2}$ satisfying $\|v_{2[t_0,\infty)}\| < \bar{\Delta}_2$, $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[t_0,\infty)}\| < \bar{\Delta}_2^u$, the output of (11) exists and satisfies, for all $t \geq t_0$,

$$(16) \qquad \|y_2(t)\| \leq \max\{\overline{\beta}_2(\|x_2(t_0)\|, t - t_0), \overline{\gamma}_2(\|v_{2[t_0,t]}\|), \overline{\gamma}_2^u(\|u_{2[t_0,t]}\|)\}.$$

Suppose that the small gain condition

$$(17) \qquad\qquad\qquad \overline{\gamma}_1 \circ \overline{\gamma}_2(r) < r, \quad r > 0$$

holds. Then the system composed of (10) and (11) with connection (12) is RISS and RIOS with restrictions $\tilde{X}_1 \times \tilde{X}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(t_0)$, u_1 and u_2 respectively, viewing $x = \mathrm{col}(x_1, x_2)$ as state, $y = \mathrm{col}(y_1, y_2)$ as output and $u = \mathrm{col}(u_1, u_2)$ as input, i.e., there exist class KL functions β and $\overline{\beta}$, class K functions γ and $\overline{\gamma}$, independent of $d(t)$, such that, for any initial state $x(t_0) \in \tilde{X}_1 \times \tilde{X}_2$, and any input functions $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[t_0,\infty)}\| < \tilde{\Delta}_1$ and $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[t_0,\infty)}\| < \tilde{\Delta}_2$, the solution and output of (10) and (11) with connection (12) exist and satisfy, for all $t \geq t_0$,

$$\|x(t)\| \leq \max\{\beta(\|x(t_0)\|, t - t_0), \gamma(\|u_{[t_0,t]}\|)\}$$
$$\|y(t)\| \leq \max\{\overline{\beta}(\|x(t_0)\|, t - t_0), \overline{\gamma}(\|u_{[t_0,t]}\|)\}$$

where,
$\gamma(s) = \max\{4\gamma_1 \circ \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s), 4\gamma_1 \circ \overline{\gamma}_1^u(s), 4\gamma_1 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 4\gamma_1 \circ \overline{\gamma}_2^u(s), 2\gamma_1^u(s), 4\gamma_2 \circ \overline{\gamma}_1 \circ$
$\overline{\gamma}_2^u(s), 4\gamma_2 \circ \overline{\gamma}_1^u(s), 4\gamma_2 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 4\gamma_2 \circ \overline{\gamma}_2^u(s), 2\gamma_2^u(s)\}$,
$\overline{\gamma}(s) = \max\{2\overline{\gamma}_1 \circ \overline{\gamma}_2^u(s), 2\overline{\gamma}_1^u(s), 2\overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 2\overline{\gamma}_2^u(s)\}$
and,
(i) If $\Delta_1, \Delta_2, \bar{\Delta}_1, \bar{\Delta}_2$ are finite,
$\tilde{X}_1 = \{x_1 \in X_1 \cap \bar{X}_1 : \overline{\beta}_1(\|x_1\|, 0) < \min\{\Delta_2, \bar{\Delta}_2\}, \overline{\gamma}_2 \circ \overline{\beta}_1(\|x_1\|, 0) < \min\{\Delta_1, \bar{\Delta}_1\}\}$
and
$\tilde{X}_2 = \{x_2 \in X_2 \cap \bar{X}_2 : \overline{\beta}_2(\|x_2\|, 0) < \min\{\Delta_1, \bar{\Delta}_1\}, \overline{\gamma}_1 \circ \overline{\beta}_2(\|x_2\|, 0) < \min\{\Delta_2, \bar{\Delta}_2\}\}$.
$\tilde{\Delta}_1 \leq \min\{\Delta_1^u, \bar{\Delta}_1^u\}, \tilde{\Delta}_2 \leq \min\{\Delta_2^u, \bar{\Delta}_2^u\}$
$s \in [0, \tilde{\Delta}_1) \Longrightarrow \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s) < \min\{\Delta_1, \bar{\Delta}_1\}, \overline{\gamma}_1^u(s) < \min\{\Delta_2, \bar{\Delta}_2\}$
and
$s \in [0, \tilde{\Delta}_2) \Longrightarrow \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s) < \min\{\Delta_2, \bar{\Delta}_2\}, \overline{\gamma}_2^u(s) < \min\{\Delta_1, \bar{\Delta}_1\}$.
(ii) If $\Delta_1, \Delta_2, \bar{\Delta}_1, \bar{\Delta}_2$ are infinite,
$\tilde{X}_1 = X_1 \cap \bar{X}_1, \tilde{X}_2 = X_2 \cap \bar{X}_2$

and
$$\tilde{\Delta}_1 \le \min\{\Delta_1^u, \overline{\Delta}_1^u\}, \ \tilde{\Delta}_2 \le \min\{\Delta_2^u, \overline{\Delta}_2^u\}.$$

Proof. First it is noted that the inequality $\overline{\gamma}_1 \circ \overline{\gamma}_2(r) < r$, $r > 0$ and the following one,

$$\overline{\gamma}_2 \circ \overline{\gamma}_1(r) < r, \quad r > 0$$

imply each other [5].

Step1: In this step, we will show that if $x_1(t_0) \in X_1 \cap \bar{X}_1$, $x_2(t_0) \in X_2 \cap \bar{X}_2$, $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[t_0,\infty)}\| < \min\{\Delta_1^u, \overline{\Delta}_1^u\}$, and $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[t_0,\infty)}\| < \min\{\Delta_2^u, \overline{\Delta}_2^u\}$, the solution of the inter-connected system exists and is bounded for all $t \ge t_0$. For this purpose, we will consider the following two cases.

(*i*) Δ_1, Δ_2, $\overline{\Delta}_1$ and $\overline{\Delta}_2$ are infinite.

Toward this end, we will first prove that the outputs y_1 and y_2 exist for all $t \ge t_0$ and are bounded in a standard way such as the proof of Theorem 10.6.1 [5]. Suppose this is not the case, for every number $R > 0$, there exists a time $T > t_0$ such that the solutions are defined on $[0, T]$ and either $\|y_1(T)\| \ge R$ or $\|y_2(T)\| \ge R$.

Without loss of generality, we only consider the case where $\|y_1(T)\| \ge R$. Choose R such that
$$R > \max\{\overline{\beta}_1(r_1, 0), \overline{\gamma}_1 \circ \overline{\beta}_2(r_2, 0), \overline{\gamma}_1^u(\Delta_1^u), \overline{\gamma}_1^u(\overline{\Delta}_1^u), \overline{\gamma}_1 \circ \overline{\gamma}_2^u(\Delta_2^u), \overline{\gamma}_1 \circ \overline{\gamma}_2^u(\overline{\Delta}_2^u)\},$$
where, $r_1 = \{x_1 \in X_1 \cap \bar{X}_1 : \sup(\|x_1\|)\}$, $r_2 = \{x_2 \in X_2 \cap \bar{X}_2 : \sup(\|x_2\|)\}$.

It follows from (15) and (16) that

(18) $\qquad \|y_{1[t_0,T]}\| \le \max\{\overline{\beta}_1(\|x_1(t_0)\|, 0), \overline{\gamma}_1(\|y_{2[t_0,T]}\|), \overline{\gamma}_1^u(\|u_{1[t_0,T]}\|)\}$

(19) $\qquad \|y_{2[t_0,T]}\| \le \max\{\overline{\beta}_2(\|x_2(t_0)\|, 0), \overline{\gamma}_2(\|y_{1[t_0,T]}\|), \overline{\gamma}_2^u(\|u_{2[t_0,T]}\|)\}.$

Substituting (19) into (18) gives that

$$\|y_{1[t_0,T]}\| \le \max\{\overline{\beta}_1(\|x_1(t_0)\|, 0), \overline{\gamma}_1 \circ \overline{\beta}_2(\|x_2(t_0)\|, 0),$$
(20) $\qquad\qquad \overline{\gamma}_1 \circ \overline{\gamma}_2(\|y_{1[t_0,T]}\|), \overline{\gamma}_1 \circ \overline{\gamma}_2^u(\|u_{2[t_0,T]}\|), \overline{\gamma}_1^u(\|u_{1[t_0,T]}\|)\}.$

Since

$$\overline{\gamma}_1 \circ \overline{\gamma}_2(\|y_{1[t_0,T]}\|) < \|y_{1[t_0,T]}\|,$$

it holds that

$$\|y_{1[t_0,T]}\| \le \max\{\overline{\beta}_1(\|x_1(t_0)\|, 0), \overline{\gamma}_1 \circ \overline{\beta}_2(\|x_2(t_0)\|, 0),$$
(21) $\qquad\qquad \overline{\gamma}_1 \circ \overline{\gamma}_2^u(\|u_{2[t_0,T]}\|), \overline{\gamma}_1^u(\|u_{1[t_0,T]}\|)\} < R$

which contradicts $\|y_1(T)\| > R$. Therefore the outputs are bounded for all $t \geq t_0$.

Since the subsystems (10) and (11) are RISS with restrictions, the solution of the inter-connected system is bounded for all $t \geq t_0$.

(ii) At least one of Δ_1, Δ_2, $\overline{\Delta}_1$ and $\overline{\Delta}_2$ is finite.

Toward this end, we will first prove that the outputs y_1 and y_2 exist for all $t \geq t_0$ and are bounded in a way motivated from the proof of Theorem 1 [20].

For any given $x(t_0) \in \tilde{X}_1 \times \tilde{X}_2$, let $p(x(t_0), \lambda)$ be a continuous path in $\tilde{X}_1 \times \tilde{X}_2$ from the origin to $x(t_0)$ with the property that $p(x(t_0), 0)$ is the origin and $p(x(t_0), 1) = x(t_0)$, and let y_1^λ and y_2^λ be the outputs starting at $x^\lambda(t_0) = p(x(t_0), \lambda)$ with inputs λu_1 and λu_2. When $\lambda = 0$, the solutions and outputs are defined on $[t_0, \infty)$ and identically zero. Note that the solutions are continuous functions of λ. Hence, for any given $T > t_0$ (arbitrarily large), $\epsilon_1 > 0$ and $\epsilon_2 > 0$, there exists λ^* such that the solution exists on $[t_0, T]$ and

(22)
$$\|y_{1[t_0,T]}^\lambda\| \leq \epsilon_1, \quad \|y_{2[t_0,T]}^\lambda\| \leq \epsilon_2$$

for all $\lambda \in [0, \lambda^*]$.

Denote that

$$\bar{\Delta}_1 = \max \left\{ \overline{\beta}_1 (\max_{\lambda \in [0,1]} \|x_1^\lambda(t_0)\|, 0), \overline{\gamma}_1 \circ \overline{\beta}_2 (\max_{\lambda \in [0,1]} \|x_2^\lambda(t_0)\|, 0), \right.$$
$$\left. \overline{\gamma}_1 \circ \overline{\gamma}_2^u (\|u_{2[t_0,\infty)}\|), \overline{\gamma}_1^u (\|u_{1[t_0,\infty)}\|) \right\},$$
$$\bar{\Delta}_2 = \max \left\{ \overline{\beta}_2 (\max_{\lambda \in [0,1]} \|x_2^\lambda(t_0)\|, 0), \overline{\gamma}_2 \circ \overline{\beta}_1 (\max_{\lambda \in [0,1]} \|x_1^\lambda(t_0)\|, 0), \right.$$
$$\left. \overline{\gamma}_2 \circ \overline{\gamma}_1^u (\|u_{1[t_0,\infty)}\|), \overline{\gamma}_2^u (\|u_{2[t_0,\infty)}\|) \right\}.$$

Since $p(x(t_0), \lambda)$ belongs to $\tilde{X}_1 \times \tilde{X}_2$ and $\|u_{1[t_0,\infty)}\| < \tilde{\Delta}_1$, $\|u_{2[t_0,\infty)}\| < \tilde{\Delta}_2$, it holds that $\bar{\Delta}_1 < \min\{\Delta_2, \overline{\Delta}_2\}$ and $\bar{\Delta}_2 < \min\{\Delta_1, \overline{\Delta}_1\}$. Let $T > t_0$ be arbitrarily large and ϵ_1, ϵ_2 satisfy $\bar{\Delta}_1 < \epsilon_1 < \min\{\Delta_2, \overline{\Delta}_2\}$, $\bar{\Delta}_2 < \epsilon_2 < \min\{\Delta_1, \overline{\Delta}_1\}$, and let $\lambda^* \in (0, 1]$ be the largest value such that (22) holds for all $\lambda \in [0, \lambda^*]$. Suppose $\lambda^* < 1$. Since $\|y_{1[t_0,T]}^\lambda\| < \min\{\Delta_2, \overline{\Delta}_2\}$ and $\|y_{2[t_0,T]}^\lambda\| < \min\{\Delta_1, \overline{\Delta}_1\}$, following the same lines as (i) when Δ_1, Δ_2, $\overline{\Delta}_1$ and $\overline{\Delta}_2$ are infinite, we have that

$$\|y_{1[t_0,T]}^\lambda\| \leq \bar{\Delta}_1 < \epsilon_1, \quad \|y_{2[t_0,T]}^\lambda\| \leq \bar{\Delta}_2 < \epsilon_2.$$

By continuity of solutions, there exists $\lambda' > \lambda^*$ such that (22) holds, contradicting that $\lambda^* < 1$. Hence $\lambda^* = 1$. Since T can be arbitrarily large, $\|y_{1[t_0,\infty)}\| < \min\{\Delta_2, \overline{\Delta}_2\}$ and $\|y_{2[t_0,\infty)}\| < \min\{\Delta_1, \overline{\Delta}_1\}$.

In both cases, the solution of the inter-connected system exist and is bounded for all $t \geq t_0$. Moreover, $\|y_{1[t_0,\infty)}\| < \min\{\Delta_2, \overline{\Delta}_2\}$ and $\|y_{2[t_0,\infty)}\| < \min\{\Delta_1, \overline{\Delta}_1\}$.

Step2: We will show the system composed of (10) and (11) is RIOS with restrictions $\tilde{X}_1 \times \tilde{X}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(t_0)$, u_1 and u_2 respectively, viewing $x = \mathrm{col}(x_1, x_2)$ as state, $y = \mathrm{col}(y_1, y_2)$ as output and $u = \mathrm{col}(u_1, u_2)$ as input.

By symmetry of y_1 and y_2, it follows from (21) that

$$\|y_{1[t_0,\infty)}\| \leq \max\left\{\overline{\beta}_1(\|x_1(t_0)\|, 0), \overline{\gamma}_1 \circ \overline{\beta}_2(\|x_2(t_0)\|, 0),\right.$$
$$\left.\overline{\gamma}_1 \circ \overline{\gamma}_2^u(\|u_{2[t_0,\infty)}\|), \overline{\gamma}_1^u(\|u_{1[t_0,\infty)}\|)\right\}$$

(23)
$$\leq \max\{\delta_1(\|x(t_0)\|), M_1\}$$

$$\|y_{2[t_0,\infty)}\| \leq \max\left\{\overline{\beta}_2(\|x_2(t_0)\|, 0), \overline{\gamma}_2 \circ \overline{\beta}_1(\|x_1(t_0)\|, 0),\right.$$
$$\left.\overline{\gamma}_2 \circ \overline{\gamma}_1^u(\|u_{1[t_0,\infty)}\|), \overline{\gamma}_2^u(\|u_{2[t_0,\infty)}\|)\right\}$$

(24)
$$\leq \max\{\delta_2(\|x(t_0)\|), M_2\}$$

where,

$$\delta_1(s) = \max\{\overline{\beta}_1(s, 0), \overline{\gamma}_1 \circ \overline{\beta}_2(s, 0)\}, \quad \delta_2(s) = \max\{\overline{\beta}_2(s, 0), \overline{\gamma}_2 \circ \overline{\beta}_1(s, 0)\},$$

$$M_1 = \max\{\overline{\gamma}_1 \circ \overline{\gamma}_2^u(\|u_{2[t_0,\infty)}\|), \overline{\gamma}_1^u(\|u_{1[t_0,\infty)}\|)\},$$

$$M_2 = \max\{\overline{\gamma}_2 \circ \overline{\gamma}_1^u(\|u_{1[t_0,\infty)}\|), \overline{\gamma}_2^u(\|u_{2[t_0,\infty)}\|)\}.$$

Hence,

$$\|y(t)\| \leq \|y_{1[t_0,\infty)}\| + \|y_{2[t_0,\infty)}\|$$
$$\leq \max\{2\delta_1(\|x(t_0)\|), 2\delta_2(\|x(t_0)\|), 2M_1, 2M_2\}$$

(25)
$$\leq \max\{\delta_3(\|x(t_0)\|), M_3\} \stackrel{def}{=} y_\infty$$

where, $\delta_3 = \max\{2\delta_1(s), 2\delta_2(s)\}$ and $M_3 = \overline{\gamma}(\|u_{[t_0,\infty)}\|)$ for any K_∞ function $\overline{\gamma}$ satisfying

$$\overline{\gamma}(s) \geq \max\{2\overline{\gamma}_1 \circ \overline{\gamma}_2^u(s), 2\overline{\gamma}_1^u(s), 2\overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 2\overline{\gamma}_2^u(s)\}.$$

Relying upon (23) and (24), the restrictions $\tilde{X}_1 \times \tilde{X}_2$ on the initial state $x(t_0)$ and $\tilde{\Delta}_1$, $\tilde{\Delta}_2$ on the inputs u_1, u_2 respectively can be computed as follows:

(*i*) If $\Delta_1, \Delta_2, \overline{\Delta}_1, \overline{\Delta}_2$ are finite,
$\tilde{X}_1 = \{x_1 \in X_1 \cap \bar{X}_1 : \overline{\beta}_1(\|x_1\|, 0) < \min\{\Delta_2, \overline{\Delta}_2\}, \overline{\gamma}_2 \circ \overline{\beta}_1(\|x_1\|, 0) < \min\{\Delta_1, \overline{\Delta}_1\}\}$
and
$\tilde{X}_2 = \{x_2 \in X_2 \cap \bar{X}_2 : \overline{\beta}_2(\|x_2\|, 0) < \min\{\Delta_1, \overline{\Delta}_1\}, \overline{\gamma}_1 \circ \overline{\beta}_2(\|x_2\|, 0) < \min\{\Delta_2, \overline{\Delta}_2\}\}.$
$\tilde{\Delta}_1 \leq \min\{\Delta_1^u, \overline{\Delta}_1^u\}$, $\tilde{\Delta}_2 \leq \min\{\Delta_2^u, \overline{\Delta}_2^u\}$
$s \in [0, \tilde{\Delta}_1) \Longrightarrow \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s) < \min\{\Delta_1, \overline{\Delta}_1\}, \overline{\gamma}_1^u(s) < \min\{\Delta_2, \overline{\Delta}_2\}$
and
$s \in [0, \tilde{\Delta}_2) \Longrightarrow \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s) < \min\{\Delta_2, \overline{\Delta}_2\}, \overline{\gamma}_2^u(s) < \min\{\Delta_1, \overline{\Delta}_1\}.$

(ii) If Δ_1, Δ_2, $\overline{\Delta}_1$, $\overline{\Delta}_2$ are infinite,
$\tilde{X}_1 = X_1 \cap \bar{X}_1$, $\tilde{X}_2 = X_2 \cap \bar{X}_2$
and
$$\tilde{\Delta}_1 \le \min\{\Delta_1^u, \overline{\Delta}_1^u\}, \quad \tilde{\Delta}_2 \le \min\{\Delta_2^u, \overline{\Delta}_2^u\}.$$

From (13) and (14), we could obtain that

$$\|x(t)\| \le \|x_1(t)\| + \|x_2(t)\|$$
$$\le \max\{\alpha_x(\|x(t_0)\|), \alpha_u(\|u_{[t_0,t]}\|), \alpha_y(\|y_{[t_0,t]}\|)\}$$
$$\le \max\{\alpha_x(\|x(t_0)\|), \alpha_u(\|u_{[t_0,\infty)}\|), \alpha_y(y_\infty)\} \stackrel{def}{=} x_\infty$$

where $\alpha_x(s) = \max\{2\beta_1(s,0), 2\beta_2(s,0)\}$, $\alpha_u(s) = \max\{2\gamma_1^u(s), 2\gamma_2^u(s)\}$, $\alpha_y(s) = \max\{2\gamma_1(s), 2\gamma_2(s)\}$.

For any time $t_1 \ge 0$, it holds that

$$\|y_1(t_0 + t_1)\| \le \max\left\{\overline{\beta}_1(\|x_1(t_0 + \frac{t_1}{2})\|, \frac{t_1}{2}), \overline{\gamma}_1(\|y_{2[t_0+\frac{t_1}{2}, t_0+t_1]}\|),\right.$$
$$\left. \overline{\gamma}_1^u(\|u_{1[t_0+\frac{t_1}{2}, t_0+t_1]}\|)\right\}$$

(26)
$$\le \max\{\overline{\beta}_1(x_\infty, \frac{t_1}{2}), \overline{\gamma}_1(\|y_{2[t_0+\frac{t_1}{2}, t_0+t_1]}\|), \overline{\gamma}_1^u(\|u_{1[t_0,\infty)}\|)\}$$

and for $\tau \in [\frac{t_1}{2}, t_1]$, it follows that

$$\|y_2(t_0 + \tau)\| \le \max\left\{\overline{\beta}_2(\|x_2(t_0 + \frac{t_1}{4})\|, \tau - \frac{t_1}{4}), \overline{\gamma}_2(\|y_{1[t_0+\frac{t_1}{4}, t_0+\tau]}\|),\right.$$
$$\left. \overline{\gamma}_2^u(\|u_{2[t_0+\frac{t_1}{4}, t_0+\tau]}\|)\right\}$$

(27)
$$\le \max\{\overline{\beta}_2(x_\infty, \frac{t_1}{4}), \overline{\gamma}_2(\|y_{1[t_0+\frac{t_1}{4}, t_0+t_1]}\|), \overline{\gamma}_2^u(\|u_{2[t_0,\infty)}\|)\}.$$

Substituting (27) into (26) gives that

$$\|y_1(t_0 + t_1)\| \le \max\{\overline{\beta}_1(x_\infty, \frac{t_1}{2}), \overline{\gamma}_1 \circ \overline{\beta}_2(x_\infty, \frac{t_1}{4}), \overline{\gamma}_1 \circ \overline{\gamma}_2(\|y_{1[t_0+\frac{t_1}{4}, t_0+t_1]}\|),$$
$$\overline{\gamma}_1 \circ \overline{\gamma}_2^u(\|u_{2[t_0,\infty)}\|), \overline{\gamma}_1^u(\|u_{1[t_0,\infty)}\|)\}$$
$$\le \max\{\tilde{\beta}_1(x_\infty, t_1), \overline{\gamma}_1 \circ \overline{\gamma}_2(\|y_{1[t_0+\frac{t_1}{4}, t_0+t_1]}\|), M_1\}$$

for any class KL function $\tilde{\beta}_1$ satisfying

(28)
$$\tilde{\beta}_1(s,t) \ge \max\{\overline{\beta}_1(s, \frac{t}{2}), \overline{\gamma}_1 \circ \overline{\beta}_2(s, \frac{t}{4})\}.$$

Denoting $z_1(t_1) = \|y_1(t_0 + t_1)\|$ gives that

(29)
$$z_1(t_1) \le \max\{\tilde{\beta}_1(x_\infty, t_1), \overline{\gamma}_1 \circ \overline{\gamma}_2(\|z_{1[\frac{t_1}{4}, t_1]}\|), M_1\}.$$

Since $\overline{\gamma}_1 \circ \overline{\gamma}_2(r) < r$ $(r > 0)$, we invoke Lemma 2.1 to conclude that there exists a class KL function $\hat{\beta}_1$ such that $z_1(t_1) \le \max\{\hat{\beta}_1(x_\infty, t_1), M_1\}$. It follows that

(30)
$$\|y_1(t)\| \le \max\{\hat{\beta}_1(x_\infty, t - t_0), M_1\}.$$

By symmetry of y_1 and y_2, there exists some class KL function $\hat{\beta}_2$ such that

$$(31) \qquad \|y_2(t)\| \leq \max\{\hat{\beta}_2(x_\infty, t - t_0), M_2\}.$$

Toward this end, consider the following two cases of y_∞ in (25).

(i) $\delta_3(\|x(t_0)\|) \geq M_3$: We have $y_\infty = \delta_3(\|x(t_0)\|)$, and $\|u_{[t_0,\infty)}\| = \overline{\gamma}^{-1}(M_3) \leq \overline{\gamma}^{-1} \circ \delta_3(\|x(t_0)\|)$. Hence,

$$x_\infty = \max\{\alpha_x(\|x(t_0)\|), \alpha_u(\|u_{[t_0,\infty)}\|), \alpha_y(y_\infty)\}$$
$$\leq \max\{\alpha_x(\|x(t_0)\|), \alpha_u \circ \overline{\gamma}^{-1} \circ \delta_3(\|x(t_0)\|), \alpha_y \circ \delta_3(\|x(t_0)\|)\} \leq \delta_4(\|x(t_0)\|)$$

for any class K function δ_4 satisfying

$\delta_4(s) \geq \max\{\alpha_x(s), \alpha_u \circ \overline{\gamma}^{-1} \circ \delta_3(s), \alpha_y \circ \delta_3(s)\}$.

As a result, (30) gives

$\|y_1(t)\| \leq \max\{\hat{\beta}_1(\delta_4(\|x(t_0)\|), t - t_0), M_1\}$.

(ii) $\delta_3(\|x(t_0)\|) < M_3$: We have $y_\infty = M_3$, then $\|y_{1[t_0,\infty)}\| \leq y_\infty = M_3$.

In both cases, we have obtained the following inequality:

$$(32) \qquad \|y_1(t)\| \leq \max\{\hat{\beta}_1(\delta_4(\|x(t_0)\|), t - t_0), M_3\}.$$

By symmetry of y_1 and y_2, we could obtain the following inequality:

$$(33) \qquad \|y_2(t)\| \leq \max\{\hat{\beta}_2(\delta_4(\|x(t_0)\|), t - t_0), M_3\}.$$

Next, we will show that the system composed of (10) and (11) is RIOS with suitable defined restrictions and gain function $\overline{\gamma}$. Combing (30) and (31) gives that

$$\|y(t)\| \leq \|y_1(t)\| + \|y_2(t)\|$$
$$\leq \max\{2\hat{\beta}_1(x_\infty, t - t_0), 2\hat{\beta}_2(x_\infty, t - t_0), 2M_1, 2M_2\}$$
$$(34) \qquad \leq \max\{\beta_3(x_\infty, t - t_0), M_3\}$$

for $\beta_3(s, t) = \max\{2\hat{\beta}_1(s, t), 2\hat{\beta}_2(s, t)\}$.

Toward this end, consider the following two cases of y_∞ in (25).

(i) $\delta_3(\|x(t_0)\|) \geq M_3$: We have $x_\infty \leq \delta_4(\|x(t_0)\|)$. As a result, (34) gives that

$$(35) \qquad \|y(t)\| \leq \max\{\beta_3(\delta_4(\|x(t_0)\|), t - t_0), M_3\}.$$

(ii) $\delta_3(\|x(t_0)\|) < M_3$: We have $y_\infty = M_3$, then $\|y(t)\| \leq M_3$.

In both cases, we have obtained the following inequality:

$$\|y(t)\| \leq \max\{\beta_3(\delta_4(\|x(t_0)\|), t - t_0), M_3\}$$
$$(36) \qquad = \max\{\overline{\beta}(\|x(t_0)\|, t - t_0), \overline{\gamma}(\|u_{[t_0,\infty)}\|)\}$$

where,

$$\overline{\beta}(s,t) = \max\{2\hat{\beta}_1(\delta_4(s),t), 2\hat{\beta}_2(\delta_4(s),t)\}.$$

Since the solution $y(t)$ depends only $u(\tau)$ on $t_0 \leq \tau \leq t$, the supremum on the right hand side of (36) can be taken over $[t_0, t]$, which yields

$$\|y(t)\| \leq \max\{\overline{\beta}(\|x(t_0)\|, t - t_0), \overline{\gamma}(\|u_{[t_0,t]}\|)\}.$$

Hence, the system composed of (10) and (11) is RIOS with restrictions $\tilde{X}_1 \times \tilde{X}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(t_0)$, u_1 and u_2 respectively, viewing $x = \mathrm{col}(x_1, x_2)$ as state, $y = \mathrm{col}(y_1, y_2)$ as output and $u = \mathrm{col}(u_1, u_2)$ as input.

Step3: We will show that the system composed of (10) and (11) is RISS with restrictions $\tilde{X}_1 \times \tilde{X}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(t_0)$, u_1 and u_2 respectively, viewing $x = \mathrm{col}(x_1, x_2)$ as state, $y = \mathrm{col}(y_1, y_2)$ as output and $u = \mathrm{col}(u_1, u_2)$ as input.

Substituting (21) into (14) gives that

$$\|x_2(t)\| \leq \max\{\beta_2(\|x_2(t_0)\|, t - t_0), \gamma_2 \circ \overline{\beta}_1(\|x_1(t_0)\|, 0), \gamma_2 \circ \overline{\gamma}_1 \circ \overline{\beta}_2(\|x_2(t_0)\|, 0),$$

$$(37) \qquad \gamma_2 \circ \overline{\gamma}_1 \circ \overline{\gamma}_2^u(\|u_{2[t_0,t]}\|), \gamma_2 \circ \overline{\gamma}_1^u(\|u_{1[t_0,t]}\|), \gamma_2^u(\|u_{2[t_0,t]}\|)\}.$$

By symmetry of x_1 and x_2, it holds that

$$\|x_1(t)\| \leq \max\{\beta_1(\|x_1(t_0)\|, t - t_0), \gamma_1 \circ \overline{\beta}_2(\|x_2(t_0)\|, 0), \gamma_1 \circ \overline{\gamma}_2 \circ \overline{\beta}_1(\|x_1(t_0)\|, 0),$$

$$(38) \qquad \gamma_1 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^u(\|u_{1[t_0,t]}\|), \gamma_1 \circ \overline{\gamma}_2^u(\|u_{2[t_0,t]}\|), \gamma_1^u(\|u_{1[t_0,t]}\|)\}.$$

Combing (37) and (38) gives that

$$(39) \qquad \|x(t)\| \leq \|x_1(t)\| + \|x_2(t)\| \leq \max\{\delta_5(\|x(t_0)\|), \tilde{\gamma}(\|u_{[t_0,\infty)}\|)\} \overset{def}{=} x_\infty'$$

where,

$$\delta_5(s,t) = \max\{2\beta_1(s,0), 2\gamma_1 \circ \overline{\beta}_2(s,0), 2\gamma_1 \circ \overline{\gamma}_2 \circ \overline{\beta}_1(s,0),$$

$$2\beta_2(s,0), 2\gamma_2 \circ \overline{\beta}_1(s,0), 2\gamma_2 \circ \overline{\gamma}_1 \circ \overline{\beta}_2(s,0)\}$$

$$\tilde{\gamma}(s) = \max\{2\gamma_1 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 2\gamma_1 \circ \overline{\gamma}_2^u(s), 2\gamma_1^u(s), 2\gamma_2 \circ \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s), 2\gamma_2 \circ \overline{\gamma}_1^u(s), 2\gamma_2^u(s)\}.$$

From (13), for any time $t_1 \geq 0$, we could obtain

$$\|x_1(t_0 + t_1)\| \leq \max\{\beta_1(\|x_1(t_0 + \frac{t_1}{2})\|, \frac{t_1}{2}), \gamma_1(\|y_{2[t_0 + \frac{t_1}{2}, t_0 + t_1]}\|), \gamma_1^u(\|u_{1[t_0 + \frac{t_1}{2}, t_0 + t_1]}\|)\}$$

$$(40) \qquad \leq \max\{\beta_1(x_\infty', \frac{t_1}{2}), \gamma_1(\|y_{2[t_0 + \frac{t_1}{2}, t_0 + t_1]}\|), \gamma_1^u(\|u_{1[t_0,\infty)}\|)\}.$$

From (33), for $\tau \in [\frac{t_1}{2}, t_1]$, it holds that

$$\|y_2(t_0 + \tau)\| \leq \max\{\hat{\beta}_2(\delta_4(\|x(t_0 + \frac{t_1}{4})\|), \tau - \frac{t_1}{4}), \overline{\gamma}(\|u_{[t_0 + \frac{t_1}{4}, t_0 + \tau]}\|)\}$$

$$(41) \qquad \leq \max\{\hat{\beta}_2(\delta_4(x_\infty'), \frac{t_1}{4}), \overline{\gamma}(\|u_{[t_0,\infty)}\|)\}.$$

Substituting (41) into (40) gives

$$\|x_1(t_0+t_1)\| \leq \max\{\beta_1(x'_\infty,\tfrac{t_1}{2}),\gamma_1\circ\hat{\beta}_2(\delta_4(x'_\infty),\tfrac{t_1}{4}),\gamma_1\circ\overline{\gamma}(\|u_{[t_0,\infty)}\|),\gamma_1^u(\|u_{1[t_0,\infty)}\|)\}$$

$$\text{(42)}\qquad \leq \max\{\beta_1^*(x'_\infty,t_1),\gamma_1^*(\|u_{[t_0,\infty)}\|)\}$$

where, $\beta_1^*(s,t)=\max\{\beta_1(s,\tfrac{t}{2}),\gamma_1\circ\hat{\beta}_2(\delta_4(s),\tfrac{t}{4})\}$, $\gamma_1^*(s)=\max\{\gamma_1\circ\overline{\gamma}(s),\gamma_1^u(s)\}$.

By symmetry of x_1 and x_2, it holds that there exist class KL function β_2^* and class K function γ_2^* such that

$$\text{(43)}\qquad\qquad \|x_2(t)\| \leq \max\{\beta_2^*(x'_\infty,t-t_0),\gamma_2^*(\|u_{[t_0,\infty)}\|)\}$$

where, $\beta_2^*(s,t)=\max\{\beta_2(s,\tfrac{t}{2}),\gamma_2\circ\hat{\beta}_1(\delta_4(s),\tfrac{t}{4})\}$, $\gamma_2^*(s)=\max\{\gamma_2\circ\overline{\gamma}(s),\gamma_2^u(s)\}$.

Combing (42) and (43) gives

$$\|x(t)\| \leq \|x_1(t)\|+\|x_2(t)\|$$

$$\text{(44)}\qquad\qquad \leq \max\{\beta^*(x'_\infty,t-t_0),\gamma^*(\|u_{[t_0,\infty)}\|)\}$$

where, $\beta^*(s,t)=\max\{2\beta_1^*(s,t),2\beta_2^*(s,t)\}$,
$\gamma^*(s)=\max\{2\gamma_1^*(s),2\gamma_2^*(s)\}=\max\{4\gamma_1\circ\overline{\gamma}_1\circ\overline{\gamma}_2^u(s),4\gamma_1\circ\overline{\gamma}_1^u(s),4\gamma_1\circ\overline{\gamma}_2\circ\overline{\gamma}_1^u(s),4\gamma_1\circ$
$\overline{\gamma}_2^u(s),2\gamma_1^u(s),4\gamma_2\circ\overline{\gamma}_1\circ\overline{\gamma}_2^u(s),4\gamma_2\circ\overline{\gamma}_1^u(s),4\gamma_2\circ\overline{\gamma}_2\circ\overline{\gamma}_1^u(s),4\gamma_2\circ\overline{\gamma}_2^u(s),2\gamma_2^u(s)\}$.

Toward this end, consider the following two cases of x'_∞ in (39).

(i) $\delta_5(\|x(t_0)\|)\geq\tilde{\gamma}(\|u_{[t_0,\infty)}\|)$: We have $x'_\infty=\delta_5(\|x(t_0)\|)$.
As a result, $\|x(t)\|\leq\max\{\beta^*(\delta_5(\|x(t_0)\|),t-t_0),\gamma^*(\|u_{[t_0,\infty)}\|)\}$.

(ii)$\delta_5(\|x(t_0)\|)<\tilde{\gamma}(\|u_{[t_0,\infty)}\|)$: We have $x'_\infty=\tilde{\gamma}(\|u_{[t_0,\infty)}\|)$.
As a result, $\|x(t)\|\leq x'_\infty=\tilde{\gamma}(\|u_{[t_0,\infty)}\|)$.
Since $\tilde{\gamma}(s)<\gamma^*(s)$ for all $s>0$, in both cases, we have obtained the following inequality

$$\text{(45)}\qquad\qquad \|x(t)\| \leq \max\{\beta(\|x(t_0)\|,t-t_0),\gamma(\|u_{[t_0,\infty)}\|)\}$$

where, $\beta(s,t)=\beta^*(\delta_5(s),t)$, $\gamma(s)=\gamma^*(s)$.
Since the solution $x(t)$ depends only on $u(\tau)$ on $t_0\leq\tau\leq t$, the supremum on the right hand side of (45) can be taken over $[t_0,t]$, which yields

$$\|x(t)\| \leq \max\{\beta(\|x(t_0)\|,t-t_0),\gamma(\|u_{[t_0,t]}\|)\}.$$

Hence, the system composed of (10) and (11) is RISS with restrictions $\tilde{X}_1\times\tilde{X}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(t_0)$, u_1 and u_2 respectively, viewing $x=\mathrm{col}(x_1,x_2)$ as state, $y=\mathrm{col}(y_1,y_2)$ as output and $u=\mathrm{col}(u_1,u_2)$ as input. This completes the proof. $\square$

REMARK 2.2.   Theorem 2.2 can be viewed as an extension of Theorem B.3.2 [6] which handles the time-invariant nonlinear systems to time-varying nonlinear systems. Nevertheless, the proof of these two theorems are quite different. The proof of Theorem B.3.2 [6] is based on the separation property for time-invariant systems. However, as mentioned in Remark 1.1, the *separation property* does not hold for the time-varying systems. As a result, we have adopted the technique of [2] and [3] to prove steps 2 and 3 of Theorem 2.2.   ☐

3. Semi-Uniform ISS Small Gain Theorem with Restrictions for Uncertain Nonlinear Time-varying Systems. In many cases, it is easier and sufficient to ascertain the semi-uniform ISS property than the ISS property for a time-varying nonlinear system. Therefore, it is interesting to infer the semi-uniform ISS property of a feedback connected system assuming each subsystem has the semi-uniform ISS property.

THEOREM 3.1.   Under Assumption 2.2, assume that subsystem (10) is semi-uniformly RISS and RALS with restrictions X_1, Δ_1 and Δ_1^u on $x_1(t_0)$, v_1 and u_1 respectively, i.e., there exist class K functions γ_1^0, γ_1, γ_1^u, $\overline{\gamma}_1^0$, $\overline{\gamma}_1$ and $\overline{\gamma}_1^u$, independent of $d(t)$, such that, for any $x_1(t_0) \in X_1$, $v_1(t) \in L_\infty^{q_1}$ satisfying $\|v_{1[t_0,\infty)}\| < \Delta_1$, $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[t_0,\infty)}\| < \Delta_1^u$, the solution and output of (10) exist and satisfy, for all $t \geq t_0$,

$$(46) \qquad \|x_1(t)\| \leq \max\{\gamma_1^0(\|x_1(t_0)\|), \gamma_1(\|v_{1[t_0,\infty)}\|), \gamma_1^u(\|u_{1[t_0,\infty)}\|)\}$$

$$(47) \qquad \|x_1\|_a \leq \max\{\gamma_1(\|v_1\|_a), \gamma_1^u(\|u_1\|_a)\}$$

$$(48) \qquad \|y_1(t)\| \leq \max\{\overline{\gamma}_1^0(\|x_1(t_0)\|), \overline{\gamma}_1(\|v_{1[t_0,\infty)}\|), \overline{\gamma}_1^u(\|u_{1[t_0,\infty)}\|)\}$$

$$(49) \qquad \|y_1\|_a \leq \max\{\overline{\gamma}_1(\|v_1\|_a), \overline{\gamma}_1^u(\|u_1\|_a)\}.$$

Also assume that subsystem (11) is semi-uniformly RISS and RALS with restrictions X_2, Δ_2 and Δ_2^u on $x_2(t_0)$, v_2 and u_2 respectively, i.e., there exist class K functions γ_2^0, γ_2, γ_2^u, $\overline{\gamma}_2^0$, $\overline{\gamma}_2$ and $\overline{\gamma}_2^u$, independent of $d(t)$, such that, for any $x_2(t_0) \in X_2$, $v_2(t) \in L_\infty^{q_2}$ satisfying $\|v_{2[t_0,\infty)}\| < \Delta_2$, $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[t_0,\infty)}\| < \Delta_2^u$, the solution and output of (11) exist and satisfy, for all $t \geq t_0$,

$$(50) \qquad \|x_2(t)\| \leq \max\{\gamma_2^0(\|x_2(t_0)\|), \gamma_2(\|v_{2[t_0,\infty)}\|), \gamma_2^u(\|u_{2[t_0,\infty)}\|)\}$$

$$(51) \qquad \|x_2\|_a \leq \max\{\gamma_2(\|v_2\|_a), \gamma_2^u(\|u_2\|_a)\}$$

$$(52) \qquad \|y_2(t)\| \leq \max\{\overline{\gamma}_2^0(\|x_2(t_0)\|), \overline{\gamma}_2(\|v_{2[t_0,\infty)}\|), \overline{\gamma}_2^u(\|u_{2[t_0,\infty)}\|)\}$$

$$(53) \qquad \|y_2\|_a \leq \max\{\overline{\gamma}_2(\|v_2\|_a), \overline{\gamma}_2^u(\|u_2\|_a)\}.$$

Suppose that the small gain condition

$$(54) \qquad\qquad\qquad \overline{\gamma}_1 \circ \overline{\gamma}_2(r) < r, \quad r > 0$$

holds. Then the system composed of (10) and (11) is semi-uniformly RISS and RALS with restrictions $\tilde{X}_1 \times \tilde{X}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(t_0)$, u_1 and u_2 respectively, viewing $x = \mathrm{col}(x_1, x_2)$ as state, $y = \mathrm{col}(y_1, y_2)$ as output and $u = \mathrm{col}(u_1, u_2)$ as input, i.e., there exist class K functions γ^0, γ^u, $\overline{\gamma}^0$ and $\overline{\gamma}^u$, independent of $d(t)$, such that, for any initial state $x(t_0) \in \tilde{X}_1 \times \tilde{X}_2$, and any input functions $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[t_0,\infty)}\| < \tilde{\Delta}_1$ and $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[t_0,\infty)}\| < \tilde{\Delta}_2$, the solution and output of (10) and (11) exist and satisfy, for all $t \geq t_0$,

(55) $$\|x(t)\| \leq \max\{\gamma^0(\|x(t_0)\|), \gamma^u(\|u_{[t_0,\infty)}\|)\}, \quad \|x\|_a \leq \gamma^u(\|u\|_a)$$

(56) $$\|y(t)\| \leq \max\{\overline{\gamma}^0(\|x(t_0)\|), \overline{\gamma}^u(\|u_{[t_0,\infty)}\|)\}, \quad \|y\|_a \leq \overline{\gamma}^u(\|u\|_a).$$

where,

$$\gamma^0(s) = \max\{2\gamma_1^0(s), 2\gamma_1 \circ \overline{\gamma}_2^0(s), 2\gamma_1 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^0(s), 2\gamma_2^0(s), 2\gamma_2 \circ \overline{\gamma}_1^0(s), 2\gamma_2 \circ \overline{\gamma}_1 \circ \overline{\gamma}_2^0(s)\}$$

$$\gamma^u(s) = \max\{2\gamma_1 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 2\gamma_1 \circ \overline{\gamma}_2^u(s), 2\gamma_1^u(s),$$
$$2\gamma_2 \circ \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s), 2\gamma_2 \circ \overline{\gamma}_1^u(s), 2\gamma_2^u(s)\},$$

$$\overline{\gamma}^0(s) = \max\{2\overline{\gamma}_1^0(s), 2\overline{\gamma}_1 \circ \overline{\gamma}_2^0(s), 2\overline{\gamma}_2^0(s), 2\overline{\gamma}_2 \circ \overline{\gamma}_1^0(s)\}$$

$$\overline{\gamma}^u(s) = \max\{2\overline{\gamma}_1 \circ \overline{\gamma}_2^u(s), 2\overline{\gamma}_1^u(s), 2\overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 2\overline{\gamma}_2^u(s)\}$$

and,

(i) If Δ_1, Δ_2 are finite,
$$\tilde{X}_1 = \{x_1 \in X_1 : \overline{\gamma}_1^0(\|x_1\|) < \Delta_2, \overline{\gamma}_2 \circ \overline{\gamma}_1^0(\|x_1\|) < \Delta_1\},$$
and
$$\tilde{X}_2 = \{x_2 \in X_2 : \overline{\gamma}_2^0(\|x_2\|) < \Delta_1, \overline{\gamma}_1 \circ \overline{\gamma}_2^0(\|x_2\|) < \Delta_2\}.$$
$\tilde{\Delta}_1 \leq \Delta_1^u$, $\tilde{\Delta}_2 \leq \Delta_2^u$
$$s \in [0, \tilde{\Delta}_1) \Longrightarrow \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s) < \Delta_1, \overline{\gamma}_1^u(s) < \Delta_2$$
and
$$s \in [0, \tilde{\Delta}_2) \Longrightarrow \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s) < \Delta_2, \overline{\gamma}_2^u(s) < \Delta_1.$$
(ii) If Δ_1, Δ_2 are infinite,
$$\tilde{X}_1 = X_1, \tilde{X}_2 = X_2$$
and
$$\tilde{\Delta}_1 \leq \Delta_1^u, \tilde{\Delta}_2 \leq \Delta_2^u.$$

Proof. The proof can be obtained by using the same technique used in the proof of Theorem 1 of [20] and is thus omitted.

□

COROLLARY 3.1. Consider the interconnection of the following two systems

(57) $$\dot{x}_1 = f_1(x_1, u_1, d, t), \quad y_1 = x_1$$

(58) $$\dot{x}_2 = f_2(x_2, v_2, u_2, d, t), \quad y_2 = x_2$$

subject to the interconnection constraint:

$$v_2 = y_1$$

where, the notations are the same as those in Theorem 3.1.

Assume that subsystem (57) is semi-uniformly RISS with restrictions X_1 and Δ_1^u on $x_1(t_0)$ and u_1 respectively, i.e., there exist class K functions γ_1^0, $\gamma_1 \equiv 0$ and γ_1^u, independent of $d(t)$, such that, for any $x_1(t_0) \in X_1$, $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[t_0,\infty)}\| < \Delta_1^u$, the solution and output of (57) exist and, for all $t \geq t_0$, (46) and (47) holds.

Also assume that subsystem (58) is semi-uniformly RISS with restrictions X_2, Δ_2 and Δ_2^u on $x_2(t_0)$, v_2 and u_2 respectively, i.e., there exist class K functions γ_2^0, γ_2 and γ_2^u, independent of $d(t)$, such that, for any $x_2(t_0) \in X_2$, $v_2(t) \in L_\infty^{q_2}$ satisfying $\|v_{2[t_0,\infty)}\| < \Delta_2$, $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[t_0,\infty)}\| < \Delta_2^u$, the solution and output of (58) exist and, for all $t \geq t_0$, (50) and (51) hold.

Then the system composed of (57) and (58) is semi-uniformly RISS with restrictions $\tilde{X}_1 \times \tilde{X}_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(t_0)$, u_1 and u_2 respectively, viewing $x = \mathrm{col}(x_1, x_2)$ as state, $y = \mathrm{col}(y_1, y_2)$ as output and $u = \mathrm{col}(u_1, u_2)$ as input, i.e., there exist class K functions γ^0, γ^u, $\overline{\gamma}^0$ and $\overline{\gamma}^u$, independent of $d(t)$, such that, for any initial state $x(t_0) \in \tilde{X}_1 \times \tilde{X}_2$, and any input functions $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_{1[t_0,\infty)}\| < \tilde{\Delta}_1$ and $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_{2[t_0,\infty)}\| < \tilde{\Delta}_2$, the solution and output of (57) and (58) exist and satisfy, for all $t \geq t_0$,

$$\|x(t)\| \leq \max\{\gamma^0(\|x(t_0)\|), \gamma^u(\|u_{[t_0,\infty)}\|)\}, \quad \|x\|_a \leq \gamma^u(\|u\|_a)$$

where,

$$\gamma^0(s) = \max\{2\gamma_1^0(s), 2\gamma_2^0(s), 2\gamma_2 \circ \overline{\gamma}_1^0(s)\}$$
$$\gamma^u(s) = \max\{2\gamma_1^u(s), 2\gamma_2^u(s), 2\gamma_2 \circ \overline{\gamma}_1^u(s)\}$$

and,
$\tilde{X}_1 = \{x_1 \in X_1 : \overline{\gamma}_1^0(\|x_1\|) < \Delta_2\}, \quad \tilde{X}_2 = X_2.$
$\tilde{\Delta}_1 \leq \Delta_1^u, \tilde{\Delta}_2 \leq \Delta_2^u$
$s \in [0, \tilde{\Delta}_1) \implies \overline{\gamma}_1^u(s) < \Delta_2.$ $\qquad\square$

4. Asymptotic Small Gain Theorem with Restrictions for Uncertain Nonlinear Time-varying Systems. In this section, we will present the asymptotic small gain theorem with restrictions for uncertain nonlinear time-varying systems. The proof is quite similar to that of Theorem 2 of [20] and is thus skipped.

THEOREM 4.1. Under Assumption 2.2, assume that both subsystems (10) and (11) are RAG and o-RAG with restrictions X_i, Δ_i and Δ_i^u on $x_i(t_0)$, v_i and u_i,

$i = 1, 2$, respectively, i.e., for $i = 1, 2$, there exist class K functions γ_i, γ_i^u, $\overline{\gamma}_i$ and $\overline{\gamma}_i^u$, independent of $d(t)$, such that, for any $x_i(t_0) \in X_i$, $v_i(t) \in L_\infty^{q_i}$ satisfying $\|v_i\|_a \leq \Delta_i$, $u_i(t) \in L_\infty^{m_i}$ satisfying $\|u_i\|_a \leq \Delta_i^u$, the solutions of (10) and (11) exist and satisfy, for all $t \geq t_0$,

$$\tag{59} \|x_i\|_a \leq \max\{\gamma_i(\|v_i\|_a), \gamma_i^u(\|u_i\|_a)\}$$

$$\tag{60} \|y_i\|_a \leq \max\{\overline{\gamma}_i(\|v_i\|_a), \overline{\gamma}_i^u(\|u_i\|_a)\}.$$

Suppose

Assumption 4.1. For all initial state in $X_1 \times X_2$ and all piecewise continuous u_1, u_2, d which are bounded on $[t_0, \infty)$, the solution of (10) and (11) with connection (12) is defined for all $t \geq t_0$;

Assumption 4.2. $\Delta_1 = \infty$;

Assumption 4.3. $\overline{\gamma}_1(\infty) < \infty$ and $\overline{\gamma}_1(\infty) \leq \Delta_2$;

Assumption 4.4. The small gain condition

$$\tag{61} \overline{\gamma}_1 \circ \overline{\gamma}_2(r) < r, \quad r > 0$$

holds.

Then, under connection (12), the system composed of (10) and (11) is RAG and o-RAG with restrictions $X_1 \times X_2$, $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ on $x(t_0)$, u_1 and u_2 respectively, viewing $x = \mathrm{col}(x_1, x_2)$ as state, $y = \mathrm{col}(y_1, y_2)$ as output and $u = \mathrm{col}(u_1, u_2)$ as input, i.e., there exist class K functions γ^u and $\overline{\gamma}^u$, independent of $d(t)$, such that, for any initial state $x(t_0) \in X_1 \times X_2$, and any input functions $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_1\|_a \leq \tilde{\Delta}_1$ and $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_2\|_a \leq \tilde{\Delta}_2$, the solution of (10) and (11) with connection (12) exists and satisfies, for all $t \geq t_0$,

$$\tag{62} \|x\|_a \leq \gamma^u(\|u\|_a),$$

$$\tag{63} \|y\|_a \leq \overline{\gamma}^u(\|u\|_a).$$

where,

$$\gamma^u(s) = \max\{2\gamma_1 \circ \overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 2\gamma_1 \circ \overline{\gamma}_2^u(s), 2\gamma_1^u(s), 2\gamma_2 \circ \overline{\gamma}_1 \circ \overline{\gamma}_2^u(s), 2\gamma_2 \circ \overline{\gamma}_1^u(s), 2\gamma_2^u(s)\},$$

$$\overline{\gamma}^u(s) = \max\{2\overline{\gamma}_1 \circ \overline{\gamma}_2^u(s), 2\overline{\gamma}_1^u(s), 2\overline{\gamma}_2 \circ \overline{\gamma}_1^u(s), 2\overline{\gamma}_2^u(s)\}$$

and $\tilde{\Delta}_2$ is such that $\tilde{\Delta}_2 \leq \Delta_2^u$,
and $\tilde{\Delta}_1$ is such that $\tilde{\Delta}_1 \leq \Delta_1^u$, and $\overline{\gamma}_1^u(\tilde{\Delta}_1) \leq \Delta_2$.

$\square$

COROLLARY 4.1.   Consider the interconnection $v_2 = y_1$ of the following two systems

$$\tag{64} \dot{x}_1 = f_1(x_1, u_1, d, t), \quad y_1 = x_1$$

$$\tag{65} \dot{x}_2 = f_2(x_2, v_2, u_2, d, t), \quad y_2 = x_2$$

where, the notations are the same as those in Theorem 4.1. Suppose:

Assumption 4.5. For all initial state in $X_1 \times X_2$ and all piecewise continuous u_1, u_2, d which are bounded on $[t_0, \infty)$, the solution of (64) and (65) with connection $v_2 = y_1$ is defined for all $t \geq t_0$;

Assumption 4.6. Subsystem (64) is RAG with restrictions X_1 and Δ_1^u on $x_1(t_0)$ and u_1 respectively;

Assumption 4.7. Subsystem (65) is RAG with restrictions X_2 and Δ_2^u on $x_2(t_0)$ and u_2 respectively.

Then system (64) and (65) with connection $v_2 = y_1$ is RAG with restrictions $X_1 \times X_2$, Δ_1^u, Δ_2^u on $(x_1(t_0), x_2(t_0))$ and u_1^u, u_2^u respectively, i.e., there exist class K function γ^u, independent of $d(t)$, such that, for any initial state $x(t_0) \in X_1 \times X_2$, and any input functions $u_1(t) \in L_\infty^{m_1}$ satisfying $\|u_1\|_a \leq \Delta_1^u$ and $u_2(t) \in L_\infty^{m_2}$ satisfying $\|u_2\|_a \leq \Delta_2^u$, the solution of (64) and (65) with connection $v_2 = y_1$ exists and satisfies, for all $t \geq t_0$,

$$\|x\|_a \leq \gamma^u(\|u\|_a)$$

where $\gamma^u(s) = \max\{2\gamma_2 \circ \gamma_1^u(s), 2\gamma_1^u(s), 2\gamma_2^u(s)\}$ and all the gain functions are defined the same way as those in Theorem 4.1.   □

The following corollary is similar to Proposition 1 in [1].

COROLLARY 4.2.   Consider the interconnections

(66) $v_{21} = y_{11}, \quad v_{22} = y_{12}, \quad v_1 = y_2$

of the following two systems

$$\Sigma_1 : \dot{x}_1 = f_1(x_1, v_1, d, t)$$
$$y_{11} = h_{11}(x_1, v_1, d, t), \quad y_{12} = h_{12}(x_1, v_1, d, t)$$
$$\Sigma_2 : \dot{x}_2 = f_2(x_2, v_{21}, v_{22}, u_2, d, t),$$
$$y_2 = h_2(x_2, v_{21}, v_{22}, u_2, d, t).$$

Suppose:

Assumption 4.8. For all initial state in $X_1 \times X_2$ and all piecewise continuous u_2, d which are bounded on $[t_0, \infty)$, the solution of Σ_1 and Σ_2 with connection (66) is defined for all $t \geq t_0$;

Assumption 4.9. Subsystem Σ_2 is RAG and o-RAG with restriction Δ_{22} on the input v_{22}, i.e., there exist class K functions γ_{21}, γ_{22}, γ_2^u, $\overline{\gamma}_{21}$, $\overline{\gamma}_{22}$ and $\overline{\gamma}_2^u$, independent of $d(t)$, such that for any initial state $x_2(t_0) \in \Re^{n_2}$ and any input $v_{22}(t)$ satisfying $\|v_{22}\|_a \leq \Delta_{22}$, the solution of Σ_2 exists and satisfies, for all $t \geq t_0$,

(67) $\|x_2\|_a \leq \max\{\gamma_{21}(\|v_{21}\|_a), \gamma_{22}(\|v_{22}\|_a), \gamma_2^u(\|u_2\|_a)\}$

(68) $\|y_2\|_a \leq \max\{\overline{\gamma}_{21}(\|v_{21}\|_a), \overline{\gamma}_{22}(\|v_{22}\|_a), \overline{\gamma}_2^u(\|u_2\|_a)\}.$

Assumption 4.10. Subsystem Σ_1 is RAG and o-RAG without restriction, i.e., there exist class K functions γ_1, $\overline{\gamma}_{11}$ and $\overline{\gamma}_{12}$, independent of $d(t)$, such that for any initial state $x_1(t_0) \in \Re^{n_1}$ and any input $v_1(t)$, the solution of Σ_1 exists and satisfies, for all $t \geq t_0$,

$$\|x_1\|_a \leq \gamma_1(\|v_1\|_a), \tag{69}$$

$$\|y_{11}\|_a \leq \overline{\gamma}_{11}(\|v_1\|_a), \tag{70}$$

$$\|y_{12}\|_a \leq \overline{\gamma}_{12}(\|v_1\|_a). \tag{71}$$

Assumption 4.11. $\overline{\gamma}_{11}(\infty) < \infty$, $\overline{\gamma}_{12}(\infty) < \infty$ and $\overline{\gamma}_{12}(\infty) \leq \Delta_{22}$

Assumption 4.12. The small gain conditions hold

$$\overline{\gamma}_{11} \circ \overline{\gamma}_{21}(r) < r, \quad \overline{\gamma}_{12} \circ \overline{\gamma}_{22}(r) < r, \quad r > 0.$$

Then under the interconnection (66), the system composed of Σ_1 and Σ_2 is RAG.

$\square$

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