

REGULARITY OF RENORMALIZED SELF-INTERSECTION LOCAL TIME FOR FRACTIONAL BROWNIAN MOTION*

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Abstract. Let B_t^H be a d -dimensional fractional Brownian motion with Hurst parameter $H \in (0, 1)$. We study the regularity, in the sense of the Malliavin calculus, of the renormalized self-intersection local time

$$\ell = \int_0^T \int_0^t \delta_0(B_t^H - B_s^H) ds dt - \mathbb{E} \left(\int_0^T \int_0^t \delta_0(B_t^H - B_s^H) ds dt \right),$$

where δ_0 is the Dirac delta function.

1. Introduction. The fractional Brownian motion on \mathbb{R}^d with Hurst parameter $H \in (0, 1)$ is a d -dimensional Gaussian process $B^H = \{B_t^H, t \geq 0\}$ with mean zero and covariance function given by

$$\mathbb{E}(B_t^{H,i} B_s^{H,j}) = \frac{\delta_{ij}}{2} (t^{2H} + s^{2H} - |t - s|^{2H}),$$

where $i, j = 1, \dots, d$, and $s, t \geq 0$. We will assume that $d \geq 2$. The *self-intersection local time* of B^H is formally defined by

$$(1) \quad I = \int_0^T \int_0^t \delta_0(B_t^H - B_s^H) ds dt,$$

where $\delta_0(x)$ is the Dirac delta function. Using the heat kernel

$$p_\varepsilon(x) = (2\pi\varepsilon)^{-d/2} e^{-\frac{|x|^2}{2\varepsilon}},$$

we approximate the self-intersection local time of B^H by

$$(2) \quad I_\varepsilon = \int_0^T \int_0^t p_\varepsilon(B_t^H - B_s^H) ds dt.$$

The asymptotic behavior of I_ε as ε tends to zero is studied in [5], and the following results are proved.

- i) If $H < \frac{1}{d}$, then I_ε converges in L^2 as ε tends to zero.
- ii) If $\frac{1}{d} < H < \frac{3}{2d}$, then

$$I_\varepsilon - TC_{H,d} \varepsilon^{-\frac{d}{2} + \frac{1}{2H}},$$

converges in L^2 as ε tends to zero to a limit ℓ , where

$$C_{H,d} = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_0^\infty \left(z^{\frac{1}{2H}} + 1 \right)^{-\frac{d}{2}} dz.$$

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iii) If $\frac{1}{d} = H < \frac{3}{2d}$, then

$$I_\varepsilon - \frac{T}{2H(2\pi)^{\frac{d}{2}}} \log \frac{1}{\varepsilon},$$

converges in L^2 as ε tends to zero.

iv) If $\frac{3}{2d} \leq H < \frac{3}{4}$, then the random variables

$$\begin{cases} \frac{1}{(\log \frac{1}{\varepsilon})^{\frac{1}{2}}} (I_\varepsilon - \mathbb{E}(I_\varepsilon)) & \text{if } H = \frac{3}{2d} \\ \varepsilon^{\frac{d}{2} - \frac{3}{4H}} (I_\varepsilon - \mathbb{E}(I_\varepsilon)) & \text{if } H > \frac{3}{2d} \end{cases}$$

converge as ε tends to zero in distribution to a normal law $N(0, T\sigma^2)$, where σ^2 is a constant depending on d and H .

We denote by ℓ the limit introduced in ii) and iii). It turns out that ℓ is also equal to the limit in L^2 of $I_\varepsilon - \mathbb{E}(I_\varepsilon)$ as ε tends to zero. If $H < \frac{1}{d}$, then ℓ will be defined as the limit in L^2 of $I_\varepsilon - \mathbb{E}(I_\varepsilon)$ as ε tends to zero. The random variable ℓ is called the *renormalized self-intersection local time* of the fractional Brownian motion.

In this paper we shall study the regularity, in the sense of Malliavin calculus, of the renormalized self-intersection local time ℓ , assuming $H < \frac{3}{2d}$. We prove that, for any real $\alpha > 0$, ℓ belongs to the Sobolev space $\mathbb{D}^{\alpha,2}$, provided $H < \min(\frac{3}{2d}, \frac{2(\alpha \wedge 1)}{d+2\alpha})$. This result generalizes that obtained by Hu in [4] in the case $\alpha = 1$. The proof of this result is established via chaos expansions.

In Section 2, we recall the chaos expansion of self-intersection local time obtained in [5]. In Section 3, we state and prove the main result of the paper.

2. Wiener chaos expansion of the self-intersection local time. In this section we recall the Wiener chaos expansion of the renormalized self-intersection local time ℓ obtained in [5].

Let \mathcal{H} be the Hilbert space defined as the closure of set \mathcal{E} of step functions from \mathbb{R}_+ to \mathbb{R}^d with respect to the scalar product

$$\langle (\mathbf{1}_{[0,t_1]}, \dots, \mathbf{1}_{[0,t_d]}), (\mathbf{1}_{[0,s_1]}, \dots, \mathbf{1}_{[0,s_d]}) \rangle_{\mathcal{H}} = \frac{1}{2^d} \prod_{i=1}^d (t_i^{2H} + s_i^{2H} - |t_i - s_i|^{2H}).$$

Then, the mapping $\mathbf{1}_{[0,t]} \rightarrow B_t^H$ is a linear isometry between \mathcal{H} and the Gaussian space spanned by B^H . For any $n \geq 1$ we denote by I_n the multiple stochastic integral which provides an isometry between the symmetric tensor product $(\mathcal{H})^{\otimes n}$ equipped with the norm $\sqrt{n!} \|\cdot\|_{\mathcal{H}^{\otimes n}}$ and the n th Wiener chaos of B^H .

Given a multi-index $\mathbf{i}_n = (i_1, \dots, i_n)$, $1 \leq i_j \leq d$, we set

$$\alpha(\mathbf{i}_n) = \mathbb{E}[X_{i_1} \cdots X_{i_n}],$$

where the X_i are independent $N(0, 1)$ random variables. Notice that

$$\alpha(\mathbf{i}_{2m}) = \frac{(2m_1)! \cdots (2m_d)!}{(m_1)! \cdots (m_d)! 2^m},$$

if $n = 2m$ is even and for each $k = 1, \dots, d$, the number of components of \mathbf{i}_{2m} equal to k , denoted by $2m_k$, is also even, and $\alpha(\mathbf{i}_n) = 0$, otherwise.

PROPOSITION 1. *Assume $Hd < \frac{3}{2}$. Then, we have*

$$\ell = \sum_{m=1}^{\infty} I_{2m}(f_{2m}),$$

where f_{2m} is the element of $(\mathcal{H})^{\otimes 2m}$ given by

$$(3) \quad \begin{aligned} f_{2m}(\mathbf{i}_{2m}, r_1, \dots, r_{2m}) &= \frac{(2\pi)^{-\frac{d}{2}} \alpha(\mathbf{i}_{2m})}{(2m)!} \\ &\times \int_0^T \int_0^t ds dt |t-s|^{-Hd-2Hm} \prod_{j=1}^{2m} \mathbf{1}_{[s,t]}(r_j). \end{aligned}$$

Let us introduce the following notation.

$$(4) \quad \lambda = |t-s|^{2H}, \quad \rho = |t'-s'|^{2H},$$

and

$$(5) \quad \mu = \frac{1}{2} [|s-t'|^{2H} + |s'-t|^{2H} - |t-t'|^{2H} - |s-s'|^{2H}].$$

Notice that λ is the variance of $B_t^{H,1} - B_s^{H,1}$, ρ is the variance of $B_{t'}^{H,1} - B_{s'}^{H,1}$, and μ is the covariance between $B_t^{H,1} - B_s^{H,1}$ and $B_{t'}^{H,1} - B_{s'}^{H,1}$, where $B^{H,1}$ denotes a one-dimensional fractional Brownian motion with Hurst parameter H .

The L^2 -norm of the $2m$ th Wiener chaos of ℓ can be computed as follows.

$$(6) \quad \begin{aligned} \mathbb{E} \left[(I_{2m}(f_{2m}))^2 \right] &= (2m)! \|f_{2m}\|_{\mathcal{H}^{\otimes(2m)}}^2 \\ &= (2m)! \sum_{m_1+\dots+m_d=m} \frac{(2m)!}{(2m_1)! \cdots (2m_d)!} \frac{(2\pi)^{-d}}{((2m)!)^2} \alpha(\mathbf{i}_{2m})^2 \\ &\quad \times \int_{\mathcal{T}} \lambda^{-\frac{d}{2}-m} \rho^{-\frac{d}{2}-m} \mu^{2m} ds dt ds' dt' \\ &= \frac{\alpha_m}{(2\pi)^d 2^{2m}} \int_{\mathcal{T}} (\lambda \rho)^{-\frac{d}{2}-m} \mu^{2m} ds dt ds' dt', \end{aligned}$$

where

$$\alpha_m = \sum_{m_1+\dots+m_d=m} \frac{(2m_1)! \cdots (2m_d)!}{(m_1!)^2 \cdots (m_d!)^2},$$

and

$$\mathcal{T} = \{(s, t, s', t') : 0 < s < t < T, 0 < s' < t' < T\}.$$

The following lemma will be useful later.

LEMMA 2. *For any $z \in [0, 1)$ we have*

$$\sum_{m=0}^{\infty} \frac{(2m)!}{(m!)^2 2^{2m}} z^m = \frac{1}{\sqrt{1-z}}.$$

Proof. This is a well-known result that can be checked, for instance, by noticing that

$$\sum_{m=0}^{\infty} \frac{(2m)!}{(m!)^2 2^{2m}} z^m = \sqrt{2\pi} \mathbb{E} \left(e^{z(Y/2)^2} \right),$$

where Y is a standard normal random variable. \square

3. Regularity of the renormalized self-intersection local time. For any $\alpha > 0$, we denote by $\mathbb{D}^{\alpha,2}$ the class of “smooth” functionals of the fractional Brownian motion, in the sense of Meyer-Watanabe. That is,

$$\mathbb{D}^{\alpha,2} = \{F \in L^2 : \sum_{n=0}^{\infty} (n+1)^\alpha \mathbb{E}((J_n(F))^2) < \infty\},$$

where $J_n(F)$ is the n -th chaos of F , namely, $F = \sum_{n=0}^{\infty} J_n(F)$.

The following theorem is the main result of this paper.

THEOREM 3. *Fix $\alpha > 0$. Assume that $H < \min(\frac{3}{2d}, \frac{2(\alpha \wedge 1)}{d+2\alpha})$. Then the renormalized self-intersection local time ℓ belongs to $\mathbb{D}^{\alpha,2}$.*

REMARK 4. *If $\alpha = 1$, we recover the result by Hu [4].*

The theorem is the direct consequence of the following two lemmas which are themselves interesting.

LEMMA 5. *a) The renormalized self-intersection local time ℓ belongs to $\mathbb{D}^{N,2}$, where $N \geq 1$, is an integer, if and only if*

$$\int_{\mathcal{T}} \mu^{2N} \delta^{-\frac{d}{2}-N} ds dt ds' dt' < \infty.$$

b) The renormalized self-intersection local time ℓ belongs to $\mathbb{D}^{N+\beta,2}$, where $N \geq 0$, is an integer, and $0 < \beta < 1$, if for some $1 > \beta' > \beta$

$$(7) \quad \int_{\mathcal{T}} \mu^{2(N+\beta')} \delta^{-\frac{d}{2}-N-\beta'} ds dt ds' dt' < \infty.$$

Proof. From (6) we obtain that, for all $\alpha > 0$, a necessary and sufficient condition for ℓ to be in $\mathbb{D}^{\alpha,2}$ is

$$(8) \quad B := \sum_{m=1}^{\infty} \frac{m^\alpha \alpha_m}{2^{2m}} \int_{\mathcal{T}} \frac{\gamma^m}{(\lambda \rho)^{\frac{d}{2}}} ds dt ds' dt' < \infty,$$

where

$$\gamma = \frac{\mu^2}{\lambda\rho}.$$

Using Lemma 2 we deduce the following formula for all $z \in [0, 1)$

$$(9) \quad \sum_{m=0}^{\infty} \frac{\alpha_m}{2^{2m}} z^m = (1-z)^{-\frac{d}{2}}.$$

Suppose first that $\alpha = N$ is an integer. In this case, differentiating both sides of (9) N times with respect to z yields

$$\sum_{m=N}^{\infty} \frac{\alpha_m}{2^{2m}} m(m-1) \cdots (m-N+1) z^{m-N} = C(1-z)^{-\frac{d}{2}-N},$$

where $C = \frac{d}{2} \left(\frac{d}{2} + 1\right) \cdots \left(\frac{d}{2} + N - 1\right)$. Hence,

$$\sum_{m=N}^{\infty} \frac{\alpha_m}{2^{2m}} m(m-1) \cdots (m-N+1) z^m = C z^N (1-z)^{-\frac{d}{2}-N},$$

and we get that (8) is equivalent to

$$\int_{\mathcal{T}} \frac{\gamma^N (1-\gamma)^{-\frac{d}{2}-N}}{(\lambda\rho)^{\frac{d}{2}}} ds dt ds' dt' = \int_{\mathcal{T}} \mu^{2N} \delta^{-\frac{d}{2}-N} ds dt ds' dt' < \infty,$$

where

$$\delta = \lambda\rho - \mu^2.$$

This proves part a) of the lemma.

Suppose now that $k = N + \beta$, with $0 < \beta < 1$, and $N \geq 0$. Multiplying both members of Equation (9) by $(y-z)^{-\beta}$ and integrating in the variable z from 0 to y , we obtain

$$\sum_{m=0}^{\infty} \frac{\alpha_m}{2^{2m}} \frac{\Gamma(1-\beta)\Gamma(m)}{\Gamma(1-\beta+m)} y^{m-\beta+1} = \int_0^y (1-z)^{-\frac{d}{2}} (y-z)^{-\beta} dz.$$

Hence,

$$\sum_{m=0}^{\infty} \frac{\alpha_m}{2^{2m}} \frac{\Gamma(1-\beta)\Gamma(m)}{\Gamma(1-\beta+m)} y^m = \int_0^1 (1-yt)^{-\frac{d}{2}} (1-t)^{-\beta} dt.$$

Differentiating this identity $N+1$ times with respect to z yields

$$\begin{aligned} & \sum_{m=N+1}^{\infty} \frac{\alpha_m}{2^{2m}} m(m-1) \cdots (m-N-2) \frac{\Gamma(1-\beta)\Gamma(m)}{\Gamma(1-\beta+m)} z^{m-N-1} \\ &= C \int_0^1 (1-zt)^{-\frac{d}{2}-N-1} t^{N+1} (1-t)^{-\beta} dt, \end{aligned}$$

where $C = \frac{d}{2} \left(\frac{d}{2} + 1 \right) \cdots \left(\frac{d}{2} + N \right)$. Hence, (8) is equivalent to

$$(10) \quad \int_{\mathcal{T}} (\lambda \rho)^{-\frac{d}{2}} \gamma^{N+1} \left(\int_0^1 (1 - \gamma y)^{-\frac{d}{2} - N - 1} y^{N+1} (1 - y)^{-\beta} dy \right) ds dt ds' dt' < \infty.$$

We claim that for all $\beta' > \beta$,

$$\int_0^1 (1 - y)^{-\beta} (1 - \gamma y)^{-\frac{d}{2} - N - 1} dy \leq k (1 - \gamma)^{-\frac{d}{2} - N - \beta'}.$$

In fact, $(1 - \gamma y)^{-\frac{d}{2} - N - 1} \leq (1 - y)^{\beta' - 1} (1 - \gamma)^{-\frac{d}{2} - N - \beta'}$. Thus,

$$\begin{aligned} \int_0^1 (1 - y)^{-\beta} (1 - \gamma y)^{-\frac{d}{2} - N - 1} dy &\leq (1 - \gamma)^{-\frac{d}{2} - N - \beta'} \int_0^1 (1 - y)^{-\beta + \beta' - 1} dy \\ &\leq \frac{1}{\beta' - \beta} (1 - \gamma)^{-\frac{d}{2} - N - \beta'}. \end{aligned}$$

Hence, (10) holds if

$$\begin{aligned} &\int_{\mathcal{T}} (\lambda \rho)^{-\frac{d}{2}} \gamma^{N+1} (1 - \gamma)^{-\frac{d}{2} - N - \beta'} ds dt ds' dt' \\ &= \int_{\mathcal{T}} (\lambda \rho)^{\beta' - 1} \mu^{2(N+1)} \delta^{-\frac{d}{2} - N - \beta'} ds dt ds' dt' < \infty, \end{aligned}$$

and (7) holds because $\mu^2 \leq \lambda \rho$.

LEMMA 6. Fix a positive real number $\alpha > 0$. Suppose that $H < \min \left(\frac{3}{2d}, \frac{2(\alpha \wedge 1)}{d + 2\alpha} \right)$. □

Then

$$\int_{\mathcal{T}} \mu^{2\alpha} \delta^{-\frac{d}{2} - \alpha} ds dt ds' dt' < \infty.$$

Proof. Denote

$$(11) \quad \mathcal{T} \cap \{s < s'\} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3,$$

where

$$\mathcal{T}_1 = \{(t, s, t', s') : 0 < s < s' < t < t' < T\},$$

$$\mathcal{T}_2 = \{(t, s, t', s') : 0 < s < s' < t' < t < T\},$$

$$\mathcal{T}_3 = \{(t, s, t', s') : 0 < s < t < s' < t' < T\}.$$

We will make use of the notation:

- i) If $(t, s, t', s') \in \mathcal{T}_1$, we put $a = s' - s$, $b = t - s'$ and $c = t' - t$. On this region, the functions λ , ρ and μ defined in (4) and (5) take the following values

$$(12) \quad \lambda = \lambda_1 := \lambda_1(a, b, c) := (a + b)^{2H}, \rho = \rho_1 := (b + c)^{2H},$$

$$(13) \quad \mu = \mu_1 := \mu_1(a, b, c) := \frac{1}{2} [(a + b + c)^{2H} + b^{2H} - c^{2H} - a^{2H}].$$

ii) If $(t, s, t', s') \in \mathcal{T}_2$, we put $a = s' - s$, $b = t' - s'$ and $c = t - t'$. On this region we will have

$$(14) \quad \lambda = \lambda_2 := b^{2H}, \rho = \rho_2 := (a + b + c)^{2H},$$

$$(15) \quad \mu = \mu_2 := \frac{1}{2} [(b + c)^{2H} + (a + b)^{2H} - c^{2H} - a^{2H}].$$

iii) If $(t, s, t', s') \in \mathcal{T}_3$, we put $a = t - s$, $b = s' - t$ and $c = t' - s'$. On this region we will have

$$(16) \quad \lambda = \lambda_3 := a^{2H}, \rho = \rho_3 := c^{2H},$$

$$(17) \quad \mu = \mu_3 := \frac{1}{2} [(a + b + c)^{2H} + b^{2H} - (b + c)^{2H} - (a + b)^{2H}].$$

For $i = 1, 2, 3$ we set

$$\delta_i = \lambda_i \rho_i - \mu_i^2.$$

Note that λ_i, ρ_i, μ_i and so on, $i = 1, 2, 3$, are functions of a, b , and c .

In the sequel we will denote by k a generic constant that may depend on H and d .

The following lower bounds were obtained by Hu in [4] using the local nondeterminism property of the fractional Brownian motion (see Berman [2]).

$$(18) \quad \delta_1 \geq k [(a + b)^{2H} c^{2H} + (b + c)^{2H} a^{2H}],$$

$$(19) \quad \delta_i \geq k \lambda_i \rho_i, \quad i = 2, 3.$$

Using the above decomposition of the region \mathcal{T} , it suffices to show that $A_i < \infty$, for $i = 1, 2, 3$, where

$$A_i := \int_{[0, T]^3} \mu_i^{2N} \delta_i^{-\frac{d}{2} - N} da db dc.$$

Then the proof of the lemma will be done in three steps:

Step 1. We claim that

$$A_1 < \infty.$$

We have

$$\begin{aligned} \mu_1 &= \frac{1}{2} ((a + b + c)^{2H} + b^{2H} - a^{2H} - c^{2H}) \\ &= \frac{1}{2} ((a^2 + b^2 + c^2 + 2ab + 2ac + 2bc)^H + b^{2H} - a^{2H} - c^{2H}) \\ &\leq b^{2H} + 2^{H-1} a^H b^H + 2^{H-1} a^H c^H + 2^{H-1} b^H c^H. \end{aligned}$$

The, using (18) yields

$$\begin{aligned}\mu_1^{2\alpha} &\leq k \left(b^{4\alpha H} + (a^{2\alpha H} b^{2\alpha H} + a^{2\alpha H} c^{2\alpha H} + b^{2\alpha H} c^{2\alpha H}) \right) \\ &\leq 3k \left(b^{2\alpha H} + \delta^\alpha \right).\end{aligned}$$

As a consequence,

$$(20) \quad \mu_1^{2\alpha} \delta_1^{-\frac{d}{2}-\alpha} \leq k \left(\delta_1^{-\frac{d}{2}} + b^{4\alpha H} \delta_1^{-\frac{d}{2}-\alpha} \right).$$

Using again (18) we obtain

$$\begin{aligned}\delta_1^{-\frac{d}{2}} &\leq k \left[(a+b)^H (b+c)^H a^H c^H \right]^{-\frac{d}{2}} \\ &\leq k (abc)^{-\frac{2}{3}Hd},\end{aligned}$$

where $-\frac{2}{3}Hd > -1$.

In order to treat the second term of (20) we consider two different cases. Assume first that $d \leq 6\alpha$. Then

$$\begin{aligned}b^{2\alpha H} \delta_1^{-\frac{d}{2}-\alpha} &\leq k \left[(a+b)^{2H} c^{2H} + (b+c)^{2H} a^{2H} \right]^{-\frac{d}{2}-\alpha} b^{4\alpha H} \\ &\leq k \left[(bc)^{2H} + (ba)^{2H} \right]^{-\frac{d}{2}-\alpha} b^{4\alpha H} \\ &\leq k (ac)^{-H(\frac{d}{2}+\alpha)} b^{H(2\alpha-d)},\end{aligned}$$

and both exponents are larger than -1 , because $H < \frac{2}{d+2\alpha} \leq \frac{1}{d-2\alpha}$.

For $d > 6\alpha$, we make use of the estimate

$$\begin{aligned}b^{2\alpha H} \delta_1^{-\frac{d}{2}-\alpha} &\leq k \left[(a+b)^H c^H (b+c)^H a^H \right]^{-\frac{d}{2}-\alpha} b^{4\alpha H} \\ &\leq k (ac)^{-(\beta_1+1)(\frac{d}{2}+\alpha)H} b^{4\alpha H - \beta_2(d+2\alpha)H},\end{aligned}$$

where $\beta_1, \beta_2 \geq 0$, and $\beta_1 + \beta_2 = 1$. Taking

$$\beta_1 = \frac{d-6\alpha}{3(d+2\alpha)}, \beta_2 = \frac{2d+12\alpha}{3(d+2\alpha)}$$

we obtain

$$b^{2\alpha H} \delta_2^{-\frac{d}{2}-\alpha} \leq k (abc)^{-\frac{2dH}{3}}.$$

Step 2. We claim that

$$A_2 < \infty.$$

If $H \geq \frac{1}{2}$ we have

$$\begin{aligned}\mu_2 &= \frac{1}{2} \left((b+c)^{2H} + (a+b)^{2H} - a^{2H} - c^{2H} \right) \\ &= Hb \int_0^1 \left[(a+bu)^{2H-1} + (c+bu)^{2H-1} \right] du \\ &\leq kb(a+b+c)^{2H-1}.\end{aligned}$$

Therefore, using (19)

$$\mu_2^{2\alpha} \delta_2^{-\frac{d}{2}-\alpha} \leq b^{-H(d+2\alpha)+2\alpha} (a+b+c)^{H(2\alpha-d)-2\alpha}.$$

Using the inequality $a+b+c \geq Ca^\beta c^\beta b^{1-2\beta}$, with $\beta = \frac{2Hd}{3Hd+6\alpha-6\alpha H}$, we obtain

$$\mu_2^{2\alpha} \delta_2^{-\frac{d}{2}-\alpha} \leq k(abc)^{-\frac{2dH}{3}}.$$

Notice that $\beta \in (0, \frac{1}{2}]$, because $H < \frac{2\alpha}{d+2\alpha}$.

Suppose now that $H < \frac{1}{2}$. In this case we have

$$\mu_2 \leq kb \left(a^{\beta(2H-1)} b^{(1-\beta)(2H-1)} + c^{\beta(2H-1)} b^{(1-\beta)(2H-1)} \right),$$

for all $\beta \in [0, 1]$. Hence,

$$\begin{aligned} \mu_2^{2\alpha} \delta_2^{-\frac{d}{2}-\alpha} &\leq k a^{\beta(2H-1)2\alpha} b^{(1-\beta)(2H-1)2\alpha+2\alpha} \delta_2^{-\frac{d}{2}-\alpha} \\ &\quad + k c^{\beta(2H-1)2\alpha} b^{(1-\beta)(2H-1)2\alpha+2\alpha} \delta_2^{-\frac{d}{2}-\alpha} \\ &=: I_1 + I_2. \end{aligned}$$

By symmetry it suffices to treat the term I_1 . We have

$$I_1 \leq k a^{\beta(2H-1)2\alpha} b^{2(1-2\beta)\alpha H+2\alpha\beta-dH} (a+b+c)^{-dH-2\alpha H}.$$

Now we make use of the lower bound

$$(a+b+c)^{-1} \geq k a^{\gamma_1} b^{\gamma_2} c^{\gamma_3},$$

where $\gamma_1 + \gamma_2 + \gamma_3 = 1$, and $\gamma_1, \gamma_2, \gamma_3 \geq 0$. In this way we obtain

$$I_1 \leq k a^{\beta_1} b^{\beta_2} c^{\beta_3},$$

where

$$\begin{aligned} \beta_1 &= \beta(2H-1)2\alpha - \gamma_1 H(d+2\alpha) \\ \beta_2 &= 2(1-2\beta)\alpha H + 2\alpha\beta - dH - \gamma_2 H(d+2\alpha) \\ \beta_3 &= -\gamma_3 H(d+2\alpha). \end{aligned}$$

If $d \leq 6\alpha$, we choose $\beta = 0$, $\gamma_1 = \gamma_3 = \frac{1}{2}$, and $\gamma_2 = 0$, and we obtain the exponents

$$\begin{aligned} \beta_1 = \beta_3 &= -\frac{H(d+2\alpha)}{2} > -1 \\ \beta_2 &= H(2\alpha-d) > -1. \end{aligned}$$

If $d > 6\alpha$, we choose

$$\beta = \frac{H(d-6\alpha)}{6(1-2H)\alpha}, \gamma_1 = \frac{d+6\alpha}{3(d+2\alpha)}, \gamma_2 = 0, \gamma_3 = \frac{2d}{3(d+2\alpha)},$$

and we obtain the exponents

$$\beta_1 = \beta_2 = \beta_3 = -\frac{2dH}{3} > -1.$$

Step 3.- We claim that

$$A_3 < \infty.$$

In this case, (17) and the inequality

$$b + vc + ua \geq k(vcu a)^\beta b^{1-2\beta},$$

with $\beta \in [0, 1]$, yield

$$\mu_3 \leq k(ac)^{1+\beta(2H-2)} b^{(1-2\beta)(2H-2)},$$

provided $\beta < \frac{1}{2(1-H)}$. As a consequence,

$$\mu_3^{2\alpha} \delta_3^{-\frac{d}{2}-\alpha} \leq k(ac)^{[1+\beta(2H-2)]2\alpha-dH-2H\alpha} b^{(1-2\beta)(2H-2)2\alpha}.$$

Choosing $\beta = \frac{6\alpha-6H\alpha-Hd}{12\alpha(1-H)}$, we obtain

$$\mu_3^{2\alpha} \delta_3^{-\frac{d}{2}-\alpha} \leq k(ac)^{-\frac{2dH}{3}}.$$

Notice that $\beta > 0$ because $H < \frac{2\alpha}{2\alpha+d} < \frac{6\alpha}{6\alpha+d}$, and also $\beta < \frac{1}{2(1-H)}$. □

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