MODULI OF BUNDLES OVER RATIONAL SURFACES AND ELLIPTIC CURVES II: NON-SIMPLY LACED CASES

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Abstract. For any non-simply laced Lie group G and elliptic curve Σ, we show that the moduli space of flat G bundles over Σ can be identified with the moduli space of rational surfaces with G-configurations which contain Σ as an anti-canonical curve. We also construct $Lie(G)$-bundles over these surfaces. The corresponding results for simply laced groups were obtained by the authors earlier in [20]. Thus we have established a natural identification for these two kinds of moduli spaces for any Lie group G.

Introduction

In [20], we constructed ADE bundles over ADE-surfaces, and established a identification for the moduli space of flat G bundles over a fixed elliptic curve Σ and the moduli space of the pairs (S, Σ) with $\Sigma \in |-K_S|$, where G is any simply laced (that is, of ADE-type), simple, compact and simply connected Lie group, and S is an ADE-surface with Σ as a smooth anti-canonical curve. This identification generalized the one for the moduli space of flat E_n bundles over Σ and the moduli space of del Pezzo surfaces of degree $9-n$ which contain Σ as an anti-canonical curve. In this paper, we construct $Lie(G)$ bundles for non-simply laced Lie group G over G-surfaces, and extend the above identification to non-simply laced cases. Therefore we establish a one-to-one correspondence between flat G bundles over a fixed elliptic curve Σ and rational surfaces with Σ as an anti-canonical curve for simple Lie groups of *all* types.

A non-simply laced Lie group G is uniquely determined by a simply laced Lie group G' and its outer automorphism group. Hence it is natural to apply the previous results for the simply laced cases to the current situation. Similar to simply-laced cases, we can define *G-surfaces* and *rational surfaces with G-configurations* (see Definition 13, 19, 26, 34). Our main result is the following theorem.

Theorem 1. *Let Σ be a fixed elliptic curve with identity $0 \in \Sigma$, G be any simple, compact, simply connected and non-simply laced Lie group. Denote $\mathcal{S}(\Sigma, G)$ the moduli space of the pairs (S, Σ), where S is a G-surface such that $\Sigma \in |-K_S|$. Denote $\mathcal{M}_\Sigma^G$ the moduli space of flat G-bundles over Σ. Then we have*

(i) $\mathcal{S}(\Sigma, G)$ can be embedded into $\mathcal{M}_\Sigma^G$ as an open dense subset.

(ii) This embedding can be extended to an isomorphism from $\overline{\mathcal{S}(\Sigma, G)}$ onto $\mathcal{M}_\Sigma^G$ by including all rational surfaces with G-configurations, and this gives us a natural and explicit compactification $\overline{\mathcal{S}(\Sigma, G)}$ of $\mathcal{S}(\Sigma, G)$.

Date: December 20, 2006.

The first author is partially supported by a RGC Grant of the Hong Kong Government.

In the following, we illustrate briefly via pictures what G-configurations and G-surfaces are in each case and compare it with the corresponding case that G' is simply-laced.

0.1. **B_n-configurations as special D_{n+1}-configurations.** In these cases we consider rational surfaces with fibration structure and a fixed smooth anti-canonical curve Σ. A B_n-configuration comes from a D_{n+1}-configuration. Roughly speaking, by saying a rational surface S has a D_{n+1}-configuration $(l_1, \cdots, l_{n+1})$, we mean that S can be considered as a blow-up of $\mathbb{F}_1$ (a *Hirzebruch surface*) at $n+1$ points on $\Sigma \in |-K_{\mathbb{F}_1}|$, such that $l_1, \cdots, l_{n+1}$ are the corresponding exceptional classes [20]. When these blown up points are *in general position*, S is called a $G = D_{n+1}$-surface. See the following picture for a surface with a D_{n+1}-configuration.

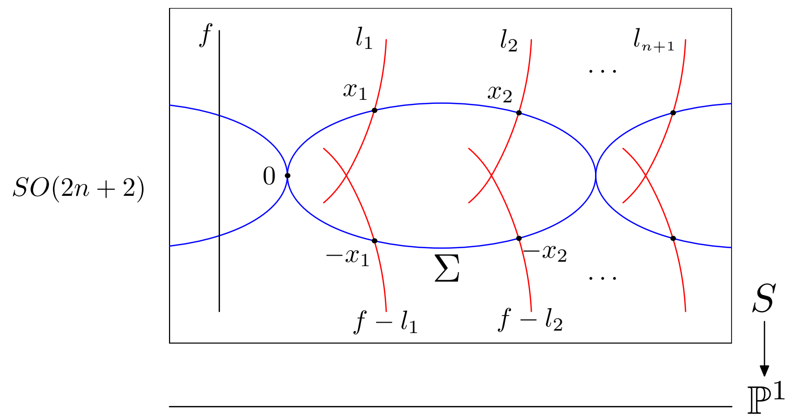

Figure 1. A surface with a D_{n+1}-configuration $(l_1, \cdots, l_{n+1})$.

Given a surface S with a D_{n+1}-configuration $\zeta = (l_1, \cdots, l_{n+1})$, if it satisfies the condition $x_1 = l_1 \cap \Sigma$ is the identity element 0 of the elliptic curve Σ, then ζ is a B_n-configuration on S (Definition 13). If all blown up points but x_1 are *in general position*, S is called a B_n-surface. See Figure 2 for a surface with a B_n-configuration.

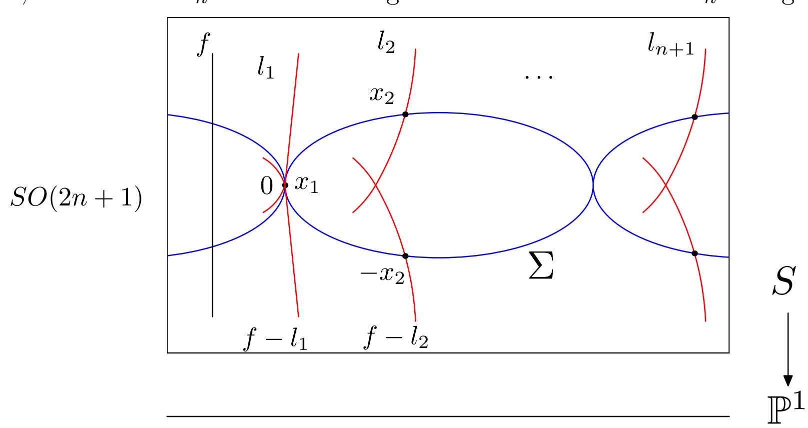

Figure 2. A surface with a B_n-configuration $(l_1, l_2, \cdots, l_{n+1})$, where $x_1 = l_1 \cap \Sigma = 0$.

0.2. **C_n-configurations as special A_{2n-1}-configurations.** In these cases, we consider rational surfaces with fibration and section structure and a fixed smooth anti-canonical curve Σ.

A C_n-configuration comes from an A_{2n-1}-configuration. By saying a rational surface S has an A_{2n-1}-configuration $(l_1, \cdots, l_{2n})$, we mean that S can be considered as a blow-up of $\mathbb{F}_1$ at $2n$ points on $\Sigma \in |-K_{\mathbb{F}_1}|$ which sum to zero, such that $l_1, \cdots, l_{2n}$ are the corresponding exceptional classes [20]. When these blown up points are *in general position*, S is called an A_{2n-1}-surface. See the following picture for a surface with an A_{2n-1}-configuration.

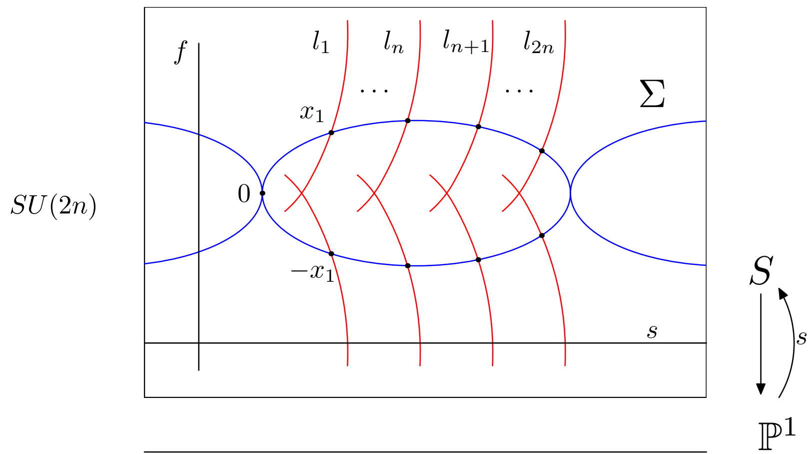

Figure 3. A surface with an A_{2n-1}-configuration $(l_1, \cdots, l_{2n})$.

Given a surface S with an A_{2n-1}-configuration $\zeta = (l_1, \cdots, l_{2n})$, if it satisfies the condition $x_i = -x_{2n+1-i}$ with $x_i = l_i \cap \Sigma$, for $i = 1, \cdots, n$, then ζ is called a C_n-configuration on S (Definition 19). If all blown up points are *in general position*, S is called a C_n-surface. See Figure 4 for a surface with a C_n-configuration.

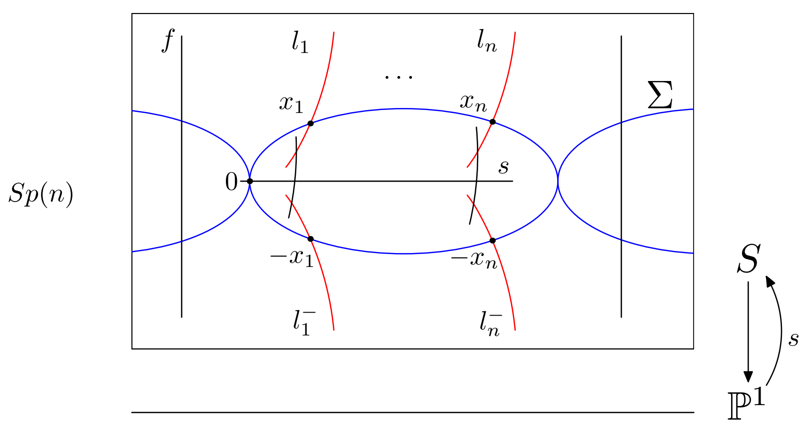

Figure 4. A surface with a C_n-configuration $(l_1, \cdots, l_n, l_n^-, \cdots, l_1^-)$.

0.3. **G_2-configurations as special D_4-configurations.** In these cases we still consider rational surfaces with fibration structure and a fixed smooth anti-canonical curve Σ.

A G_2-configuration comes from a D_4-configuration. We have seen what a D_4-configuration is from Subsection 0.1. Roughly speaking, by saying a rational surface S has a D_4-configuration $(l_1, \cdots, l_4)$, we mean that S can be considered as a blow-up of $\mathbb{F}_1$ at 4 points on $\Sigma \in |-K_{\mathbb{F}_1}|$, such that $l_1, \cdots, l_4$ are the corresponding exceptional classes [20]. When these blown up points are *in general position*, S is called a $G = D_4$-surface. See Figure 5 for a surface with a D_4-configuration.

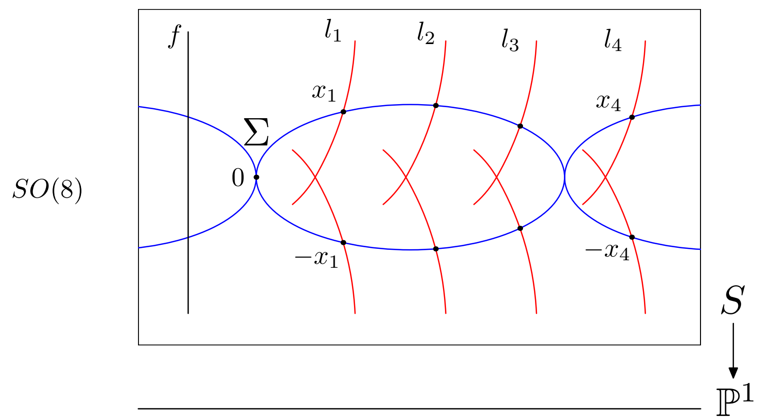

Figure 5. A surface with a D_4-configuration $(l_1, \cdots, l_4)$.

Given a surface S with a D_4-configuration $\zeta = (l_1, \cdots, l_4)$, if it satisfies these two conditions $x_1 = 0$ and $x_4 = x_2 + x_3$, where $x_i = l_i \cap \Sigma$, then ζ is called a G_2-configuration on S (Definition 26). If all blown up points but x_1 are *in general position*, S is called a G_2-surface. See Figure 6 for a surface with a G_2-configuration.

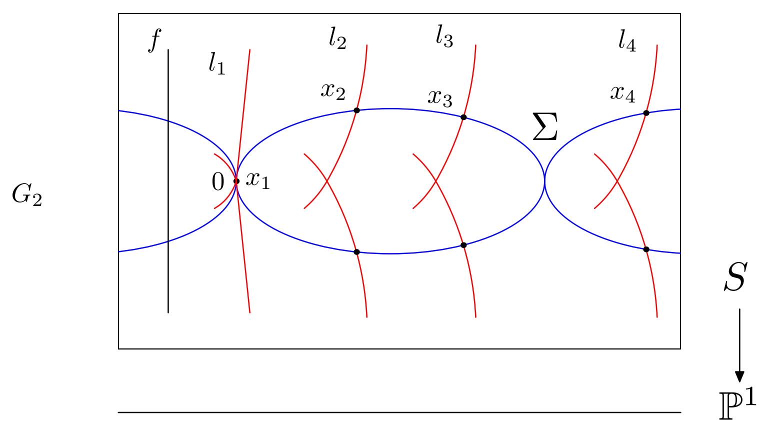

Figure 6. A surface with a G_2-configuration (l_1, l_2, l_3, l_4),
where $x_1 = 0$ and $x_4 = x_2 + x_3$ with $x_i = l_i \cap \Sigma$.

0.4. **F_4-configurations as special E_6-configurations.** In these cases we consider rational surfaces which are blow-ups of the projective plane $\mathbb{P}^2$ at 6 points in almost general position, and which contain a fixed smooth anti-canonical curve Σ [20].

An F_4-configuration comes from an E_6-configuration. Recall that by saying a rational surface S has an E_6-configuration $(l_1, \cdots, l_6)$, we mean that S can be considered as a blow-up of $\mathbb{P}^2$ at 6 points on $\Sigma \in |-K_{\mathbb{P}^2}|$, such that $l_1, \cdots, l_6$ are the corresponding exceptional classes. When these blown up points are *in general position*, S is called an E_6-surface, which is in fact a cubic surface. See Figure 7 for a surface with an E_6-configuration.

E_6

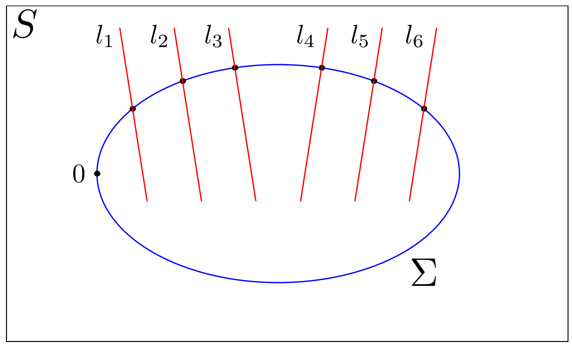

Figure 7. A surface with an E_6-configuration $(l_1, \cdots, l_6)$,

Given a surface S with an E_6-configuration $\zeta = (l_1, \cdots, l_6)$, if it satisfies the condition $x_1 + x_6 = x_2 + x_5 = x_3 + x_4$, where $x_i = l_i \cap \Sigma$, then ζ is called an F_4-configuration on S (Definition 34). If all blown up points are *in general position*, S is called an F_4-surface. See Figure 8 for a surface with an F_4-configuration.

F_4

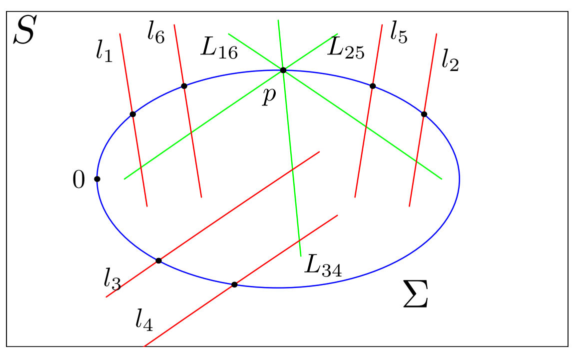

Figure 8. A surface with an F_4-configuration $(l_1, \cdots, l_6)$, where three lines L_{16}, L_{25}, L_{34} meet at $p \in \Sigma$, or equivalently, $x_1 + x_6 = x_2 + x_5 = x_3 + x_4$ with $x_i = l_i \cap \Sigma$.

Moreover, we can construct $\mathcal{G}$= Lie(G) bundles over S with a G-configuration. By restriction, we obtain Lie(G) bundles over Σ. And we can also constructed some natural fundamental representation bundles over Σ which have interesting geometric meanings, such that the Lie algebra bundles are the automorphism bundles of these representation bundles preserving certain algebraic structures.

Notation 2. In this paper, the notations are the same as those in [20]. Let G be a compact, simple and simply-connected Lie group. We denote

$r(G)$: the rank of G;
$R(G)$: the root system;
$R_c(G)$: the coroot system;
$W(G)$: the Weyl group;
$\Lambda(G)$: the root lattice;
$\Lambda_c(G)$: the coroot lattice;
$\Lambda_w(G)$: the weight lattice;
$T(G)$: a maximal torus;
$ad(G)$: the adjoint group of G, i.e. $G/C(G)$ where $C(G)$ is the center of G;
$\Delta(G)$: a simple root system of G.
$Out(G)$: the outer automorphism group of G, which is defined as the quotient of the automorphism group of G by its inner automorphism group. It is well-known that $Out(G)$ is isomorphic to the diagram automorphism group of the Dynkin diagram of G.

When there is no confusion, we just ignore the letter G.

1. Reductions to simply laced cases

From now on, we always assume that G is a compact, simple, simply-connected Lie group of non-simply laced type, that is, of type B_n, C_n, F_4, G_2. There are two natural approaches to reduce situations to simply laced cases. One is embedding G into a simply laced Lie group G' such that G is the subgroup fixed by the outer automorphism group of G'. Another is taking the simply laced subgroup G'' of maximal rank.

In the following we explain the first reduction. The following result is well-known.

Proposition 3. *Let G be a compact, non-simply laced, simple, and simply connected Lie group. There exists a simple, simply connected and simply laced Lie group G', s.t. $G \subset G'$ and $G = (G')^{\rho}$, where ρ is an outer automorphism of G' of order 3 for $G' = D_4$, and of order 2 otherwise.*

Proof. By the functorial property, we just need to prove it in the Lie algebra level. For the construction of $\mathcal{G} = Lie(G)$ and $\mathcal{G}' = Lie(G')$, one can see [17] for the details, where the construction of Lie algebras is determined by the construction of root systems. □

Remark 4. For later use, we list the construction of non-simply laced root systems via simply laced root systems.

(1) $G = C_n = Sp(n)$, $G' = A_{2n-1} = SU(2n)$.
$\Delta(G') = \{\alpha_i, i = 1, \cdots, 2n-1\}$.
$Out(G') = \{1, \rho\} \cong \mathbb{Z}_2$, *where* $\rho(\alpha_i) = \alpha_{2n-i}, i = 1, \cdots, n-1$, *and* $\rho(\alpha_n) = \alpha_n$.
$\Delta(G) = \{\beta_i = \frac{1}{2}(\alpha_i + \alpha_{2n-i}), i = 1, \cdots, n-1, \beta_n = \alpha_n\}$.

(2) $G = B_n = Spin(2n+1)$, $G' = D_{n+1} = Spin(2n+2)$.
$\Delta(G') = \{\alpha_i, i = 1, \cdots, n+1\}$.
$Out(G') = \{1, \rho\} \cong \mathbb{Z}_2$, *where* $\rho(\alpha_i) = \alpha_i, i = 3, \cdots, n+1$, $\rho(\alpha_1) = \alpha_2$, $\rho(\alpha_2) = \alpha_1$.
$\Delta(G) = \{\beta_1 = \frac{1}{2}(\alpha_1 + \alpha_2), \beta_i = \alpha_{i+1}, i = 2, \cdots, n\}$.

(3) $G = F_4$, $G' = E_6$.
$\Delta(G') = \{\alpha_i, i = 1, \cdots, 6\}$.
$Out(G') = \{1, \rho\} \cong \mathbb{Z}_2$, *where* $\rho(\alpha_i) = \alpha_{6-i}, i = 1, \cdots, 5$, *and* $\rho(\alpha_6) = \alpha_6$.
$\Delta(G) = \{\beta_1 = \frac{1}{2}(\alpha_1 + \alpha_5), \beta_2 = \frac{1}{2}(\alpha_2 + \alpha_4), \beta_3 = \alpha_3, \beta_4 = \alpha_6\}$.

(4) $G = G_2$, $G' = D_4 = Spin(8)$.
$\Delta(G') = \{\alpha_i, i = 1, \cdots, 4\}$.
$Out(G') = \langle \rho_1, \rho_2 \rangle \cong S_3$, *where* ρ_1 *interchanges* α_1 *and* α_2, *and* ρ_2 *interchanges* α_1 *and* α_4.
$\Delta(G) = \{\beta_1 = \frac{1}{3}(\alpha_1 + \alpha_2 + \alpha_4), \beta_2 = \alpha_3\}$.

The Dynkin diagrams of G and G' are as the following:

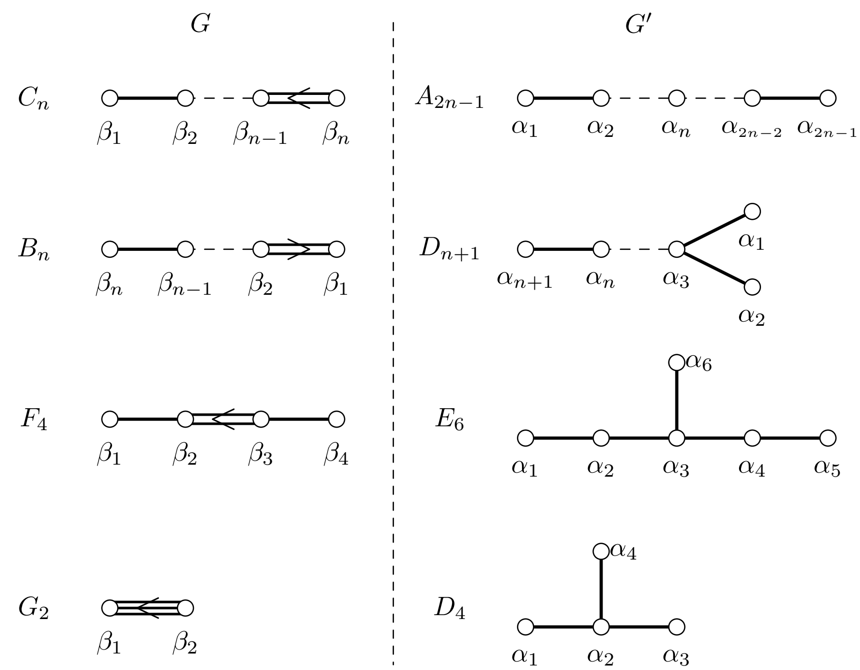

Figure 9. Non-simply laced G reduced to simply laced G'.

Remark 5. Note that $W(G)$ is the subgroup of $W(G')$ fixing the root system $R(G)$, and also the subgroup pointwise fixed by $Out(G')$. For a root α, let $S_\alpha \in W(G)$ be the reflection with respect to α, that is, $S_\alpha(x) = x + (x, \alpha)\alpha$. Thus as a subgroup of $W(A_{2n-1})$, $W(C_n)$ is generated by $S_{\alpha_i} \circ S_{\alpha_{2n-i}}$ for $i = 1, \cdots, n-1$ and S_{α_n}. As a subgroup of $W(D_{n+1})$, $W(B_n)$ is generated by $S_{\alpha_1} \circ S_{\alpha_2}$ and S_{α_i} for $i = 3, \cdots, n+1$. As a subgroup of $W(E_6)$, $W(F_4)$ is generated by $S_{\alpha_1} \circ S_{\alpha_5}$, $S_{\alpha_2} \circ S_{\alpha_4}$, S_{α_3} and S_{α_6}. As a subgroup of $W(D_4)$, $W(G_2)$ is generated by $S_{\alpha_1} \circ S_{\alpha_2} \circ S_{\alpha_4}$ and S_{α_3}.

In the following we let Σ be a fixed elliptic curve with identity element 0, and we fix a primitive d^{th} root of $\Sigma \cong Jac(\Sigma)$, where $d = 2$ for D_n case, $d = 9 - n$ for E_n case, and $d = n + 1$ for A_n case, respectively (see [20]). Recall that for any compact, simple and simply-connected Lie group H, the moduli space of flat H bundles over Σ is

$$\mathcal{M}_\Sigma^H \cong (\Lambda_c(H) \otimes \Sigma)/W(H).$$

For G', the group $Out(G')$ acts on

$$(\Lambda_c(G') \otimes \Sigma)/W(G')$$

naturally.

Let χ be the natural map from $(\Lambda_c(G) \otimes \Sigma)/W(G)$ to the fixed part

$$((\Lambda_c(G') \otimes \Sigma)/W(G'))^{Out(G')}.$$

The image of χ is contained in a connected component of the fixed part.

Lemma 6. *The map*

$$\chi : (\Lambda_c(G) \otimes \Sigma)/W(G) \to ((\Lambda_c(G') \otimes \Sigma)/W(G'))^{Out(G')}$$

is injective.

Proof. It suffices to prove that for any $x, y \in \Lambda(G) \otimes \Sigma$, if $\exists\ w' \in W(G')$, such that $w'(x) = y$, then $\exists\ w \in W(G)$, such that $w(x) = y$. For A_n and D_n cases, this is obvious if we check the root lattices. For E_6 case, we can also check it directly with the help of computer. Of course we can also check this case by hand following the discussion in Section 2.4.1. □

Corollary 7. *(i) The fixed part $((\Lambda_c(G') \otimes \Sigma)/W(G'))^{Out(G')}$ is determined by the condition $\rho(x) = x$, up to $W(G')$-action, where $x \in \Lambda_c(G') \otimes \Sigma$, and ρ is a generator of $Out(G')$, of order 3 for $G' = D_4$ and order 2 for $G' = A_n,\ E_n$.*

(ii) The moduli space $\mathcal{M}_\Sigma^G \cong (\Lambda_c(G) \otimes \Sigma)/W(G)$ is a connected component of the fixed part

$$(\mathcal{M}_\Sigma^{G'})^{Out(G')} \cong ((\Lambda_c(G') \otimes \Sigma)/W(G'))^{Out(G')}$$

containing the trivial G' bundle.

Proof. (i) For any $x \in \Lambda_c(G') \otimes \Sigma$, denote $\bar{x}$ the class in $(\Lambda_c(G') \otimes \Sigma)/W(G')$. Then $\rho(\bar{x}) = \bar{x}$ if and only if there exists $w \in W(G')$, such that $\rho(x) = w(x)$. Thus $w^{-1}\rho(x) = x$. But $w^{-1}\rho \in Out(G')$ since $Out(G') = Aut(G')/W(G')$. Thus we can take a new simple root system such that $w^{-1}\rho$ is the generator of the diagram automorphism (the automorphism of order 3 for D_4).

(ii) By (i), $(\Lambda_c(G) \otimes \Sigma)/W(G)$ and $(\mathcal{M}_\Sigma^{G'})^{Out(G')}$ are both orbifolds with the same dimension. Thus the result follows from Lemma 6. □

If we express the moduli space of flat G bundles over Σ as $(T \times T)/W(G)$, where T is a maximal torus of G, then we have the following corollary.

Corollary 8. *If two elements of $T \times T$ are conjugate under $W(G')$, then they are also conjugate under $W(G)$.* □

Another method is to reduce G to its simply-laced subgroup G'' of maximal rank, and apply the results for simply laced cases to current situation. In another occasion we will discuss our moduli space of G-bundles from this aspect in detail. Here we just mention the following well-known fact from Lie theory.

Proposition 9. *There exists canonically a simply laced Lie subgroup G'' of G, which is of maximal rank, that is, G'' and G share a common maximal torus. And there is a short exact sequence*

$$1 \to W(G'') \to W(G) \to Out(G'') \to 1,$$

where $Out(G'')$ is the outer automorphism group of G''. Thus, if we write the moduli space as $\mathcal{M}_\Sigma^G = (T \times T)/W$, then

$$\mathcal{M}_\Sigma^G = \mathcal{M}_\Sigma^{G''}/Out(G''). \qquad \square$$

Remark 10. We give this construction of G'' in each case.

(1) For $G = Sp(n)$, $G'' = SU(2)^n$. $Out(G'')$ is the group S_n of permutations of the n copies of $SU(2)$ in G''.

(2) For $G = G_2$, $G'' = SU(3)$. $Out(G'')$ is the group $\mathbb{Z}_2$ that exchanges the 3-dimensional representation of $SU(3)$ with its dual.

(3) For $G = Spin(2n+1)$, $G'' = Spin(2n)$. $Out(G'')$ is the group $\mathbb{Z}_2$ that exchanges the two spin representations of $Spin(2n)$.

(4) For $G = F_4$, $G'' = Spin(8)$. $Out(G'')$ is the triality group S_3 that permutes the three 8-dimensional representations of $Spin(8)$.

2. Flat G bundles over elliptic curves and rational surfaces: non-simply laced cases

In this section, we study case by case the G bundles over elliptic curves and rational surfaces for a non-simply laced Lie group G.

2.1. **The $B_n(n \geq 2)$ bundles.** According to the arguments of last section, for $G = Spin(2n+1)$ we can take $G' = Spin(2n+2)$, such that $G = (G')^{Out(G')}$.

Let $S = Y_{n+1}$ be a rational surface with a D_{n+1}-configuration [20] which contains Σ as a smooth anti-canonical curve. Recall [20] that Y_{n+1} is a blow-up of $\mathbb{F}_1$ at $n+1$ points $x_1, \cdots, x_{n+1}$ on Σ, with corresponding exceptional classes $l_1, \cdots, l_{n+1}$. Let f be the class of fibers in $\mathbb{F}_1$, and s be the section such that $0 = s \cap \Sigma$ is the identity element of Σ. The Picard group of Y_{n+1} is $H^2(Y_{n+1}, \mathbb{Z})$, which is a lattice with basis $s, f, l_1, \cdots, l_{n+1}$. The canonical line bundle $K = -(2s + 3f - \sum_{i=1}^{n+1} l_i)$.

We know from [20] that

$$P_{n+1} := \{x \in H^2(Y_{n+1}, \mathbb{Z}) \mid x \cdot K = x \cdot f = 0\}$$

is a root lattice of D_{n+1} type. We take a simple root system of G' as

$$\Delta(D_{n+1}) = \{\alpha_1 = l_1 - l_2, \alpha_2 = f - l_1 - l_2, \alpha_3 = l_2 - l_3, \cdots, \alpha_{n+1} = l_n - l_{n+1}\}.$$

Let ρ be the generator of $Out(G') \cong \mathbb{Z}_2$, such that $\rho(\alpha_1) = \alpha_2, \rho(\alpha_2) = \alpha_1$ and $\rho(\alpha_i) = \alpha_i$ for $i = 3, \cdots, n+1$.

From [20] we know that the pair (S, Σ) determines a homomorphism

$$u \in Hom(\Lambda(G'), \Sigma)$$

which is given by the restriction map:

$$u(\alpha) = \mathcal{O}(\alpha)|_\Sigma.$$

Lemma 11. *Let $u \in Hom(\Lambda(G'), \Sigma)$ correspond to a pair (S, Σ), where S is a surface with a D_{n+1}-configuration. Then $\rho \cdot u = u$ if and only if $2x_1 = 0$.*

Proof. Since u is the restriction map: $\alpha_i \mapsto \mathcal{O}(\alpha_i)|_\Sigma$, $u(\alpha_1) = \mathcal{O}(l_1 - l_2)|_\Sigma = x_1 - x_2$, and $u(\alpha_2) = \mathcal{O}(f - l_1 - l_2)|_\Sigma = -x_1 - x_2$. Hence $\rho \cdot u = u \Leftrightarrow u(\alpha_1) = u(\alpha_2) \Leftrightarrow x_1 - x_2 = -x_1 - x_2 \Leftrightarrow 2x_1 = 0 \Leftrightarrow x_1$ is one of the 4 points of order 2 on the elliptic curve Σ. $\square$

As in [20], we denote $\mathcal{S}(\Sigma, G')$ the moduli space of $G' = D_{n+1}$-surfaces with a fixed anti-canonical curve Σ, and $\overline{\mathcal{S}(\Sigma, G')}$ the natural compactification by including all rational surfaces with D_{n+1}-configurations (Figure 1). From [20] we know that $\phi : \overline{\mathcal{S}(\Sigma, G')} \xrightarrow{\sim} \mathcal{M}_\Sigma^{G'}$ is an isomorphism.

Corollary 12. *For $u \in \mathcal{M}_\Sigma^G \hookrightarrow (\mathcal{M}_\Sigma^{G'})^{Out(G')}$, $\phi^{-1}(u) \in \overline{\mathcal{S}(\Sigma, G')}$ represents a class of surfaces $Y_{n+1}(x_1, \cdots, x_{n+1})$ with $x_1 = 0$, and such a surface corresponds to a boundary point in the moduli space, that is, $\phi^{-1}(u) \in \overline{\mathcal{S}(\Sigma, G')} \backslash \mathcal{S}(\Sigma, G')$.*

Proof. By Lemma 11, $u \in (\mathcal{M}_\Sigma^{G'})^{Out(G')}$ if and only if $2x_1 = 0$. There are 4 connected components corresponding to 4 points of order 2 on Σ. Since $\mathcal{M}_\Sigma^G$ is the component containing the trivial G' bundle, we have $x_1 = 0$. Recall (§4, [20]) that $Y_{n+1}(x_1, \cdots, x_{n+1}) \in \mathcal{S}(\Sigma, G')$ if and only if $0, x_1, \cdots, x_{n+1}$ are in general position, which implies in particular $x_1 \neq 0$. Hence $\phi^{-1}(u)$ corresponds to a boundary point. $\square$

Denote $S = Y'_{n+1}(x_1 = 0, x_2, \cdots, x_{n+1})$ (or Y'_{n+1} for brevity) the blow-up of $\mathbb{F}_1$ at $n+1$ points $x_1 = 0, x_2, \cdots, x_{n+1}$ on Σ, with exceptional divisors $l_1, l_2, \cdots, l_{n+1}$, where $\Sigma \in |-K_S|$. Similar to the simply laced cases, we give the following definition.

Definition 13. A B_n*-exceptional system* on S is an n-tuple $(e_1, e_2, \cdots, e_{n+1})$ where e_i's are exceptional divisors such that $e_i \cdot e_j = 0 = e_i \cdot f, i \neq j$ and $y_1 = e_1 \cap \Sigma = 0$ is the identity of Σ. A B_n*-configuration* on S is a B_n-exceptional system $\zeta_{B_n} = (e_1, e_2, \cdots, e_{n+1})$ such that we can consider S as a blow-up of $\mathbb{F}_1$ at $n+1$ points $y_1 = 0, y_2, \cdots, y_{n+1}$ on Σ, that is $S = Y'_{n+1}(y_1 = 0, y_2, \cdots, y_{n+1})$, with corresponding exceptional divisors $e_1, e_2, \cdots, e_{n+1}$. When S has a B_n-configuration, we call S a *(rational) surface with a* B_n*-configuration* (see Figure 2).

When $x_2, \cdots, x_{n+1} \in \Sigma$ with $x_i \neq 0$ for all i are in general position (refer to §4 of [20] for definition), any B_n-exceptional system on S consists of exceptional curves. Such a surface is called a B_n*-surface.* So a B_n-surface must have a B_n-configuration.

Lemma 14. *(i) Let S be a rational surface with a B_n-configuration. Then the Weyl group $W(B_n)$ acts on all B_n-exceptional systems on S simply transitively.*

(ii) Let S be a B_n-surface. Then the Weyl group $W(B_n)$ acts on all B_n-configurations simply transitively.

Proof. It suffices to prove (i). Let $(e_1, e_2, \cdots, e_{n+1})$ be a B_n-exceptional system on S. By Definition 13, $e_i = l_{\sigma(i)}$ or $f - l_{\sigma(i)}$ for $i \neq 1$, where σ is a permutation of $\{2, \cdots, n+1\}$. Note that according to Remark 5, the Weyl group $W(B_n)$ acts as the group generated by permutations of the n pairs $\{(l_i, f - l_i) \mid i = 2, \cdots, n+1\}$

and interchanging of l_i and $f-l_i$ in each pair $(l_i, f-l_i)_{i\geq 2}$. Then the result follows. $\square$

Let $\mathcal{S}(\Sigma, B_n)$ be the moduli space of pairs (S, Σ) where S is a B_n-surface (so the blown-up points $x_1 = 0, x_2, \cdots, x_{n+1}$ are in general position), and $\Sigma \in |-K_S|$. Denote $\mathcal{M}_{\Sigma}^{B_n}$ the moduli space of flat B_n bundles over Σ. Then applying Corollary 12 we have the following identification.

Proposition 15. *(i) $\mathcal{S}(\Sigma, B_n)$ is embedded into $\mathcal{M}_{\Sigma}^{B_n}$ as an open dense subset.*

(ii) Moreover, this embedding can be extended naturally to an isomorphism

$$\overline{\mathcal{S}(\Sigma, B_n)} \cong \mathcal{M}_{\Sigma}^{B_n},$$

by including all rational surfaces with B_n-configurations.

Proof. The proof is similar to that in ADE cases [20]. Firstly, we have $\mathcal{M}_{\Sigma}^{B_n} \cong \Lambda_c(B_n) \otimes_{\mathbb{Z}} \Sigma / W(B_n)$, and $\Lambda_c(B_n) \otimes_{\mathbb{Z}} \Sigma / W(B_n) \cong Hom(\Lambda(B_n), \Sigma)/W(B_n)$ when we fixed the square root of unity of $Jac(\Sigma) \cong \Sigma$. Refer to Section 3 of [20] for the detail.

Secondly, the restriction from S to Σ induces a map (again denoted by ϕ)

$$\phi : \mathcal{S}(\Sigma, B_n) \to Hom(\Lambda(B_n), \Sigma)/W(B_n).$$

This map is well-defined, since by Lemma 14, choosing and fixing a B_n-configuration on S is equivalent to choosing and fixing a system of simple roots $\Delta(B_n)$.

Thirdly, the map ϕ is injective. For this, we take a simple root system of B_n as

$$\beta_1 = f - 2\ l_2 \text{ and } \beta_k = 2\ \alpha_{k+1} \text{ for } 2 \leq k \leq n.$$

Then the restriction induces an element $u \in Hom(\Lambda(B_n), \Sigma)$, which satisfies the following system of linear equations

$$\begin{cases} -2\ x_2 = p_1, \\ 2(x_k - x_{k+1}) = p_k,\ k = 2, \cdots, n. \end{cases}$$

where $p_i = u(\beta_i)$. Obviously, the solution of this system of linear equations exists uniquely for given p_i with $1 \leq i \leq n$.

Finally, the statement (ii) comes from Corollary 12 and the existence of the solutions to the above system of linear equations. $\square$

Remark 16. The situation here is very similar to that in the compactification theory of the moduli space of (projective) $K3$ surfaces. A natural question there is how to extend the global Torelli theorem to the boundary components of a compactification [9][18][25][5]. If we consider the map $\phi : \mathcal{S}(\Sigma, G) \to \mathcal{M}_{\Sigma}^{G}$ [20] for $G = A_n, D_n$ or E_n as a type of period map, then the main result of [20] is a type of global Torelli theorem. And Proposition 15 implies that we can extend the theorem of Torelli type in D_{n+1} case to a boundary component of the natural compactification.

In the following, we let $S = Y_{n+1}(x_1, \cdots, x_{n+1})$ be the blow-up of $\mathbb{F}_1$ at $n+1$ points. We can construct a Lie algebra bundle on S. Here we don't need the existence of the anti-canonical curve Σ. According to Section 2, we have a root system of B_n type consisting of divisors on S:

$$R(B_n) \triangleq \{\pm(f - 2\ l_i), 2(l_i - l_j), \pm 2(f - l_i - l_j)\ \mid\ i \neq j, 2 \leq i, j \leq n+1\}.$$

Thus we can construct a Lie algebra bundle of B_n-type over S:

$$\mathscr{B}_n \triangleq \mathcal{O}^{\bigoplus n} \bigoplus_{D\in R(B_n)} \mathcal{O}(D).$$

The fiberwise Lie algebra structure of $\mathscr{B}_n$ is defined as follows (the argument here is the same as that in [20]).

Fix the system of simple roots of R_n as

$$\Delta(B_n) = \{\alpha_1 = f - 2l_2, \alpha_2 = 2(l_2 - l_3), \cdots, \alpha_n = 2(l_n - l_{n+1})\},$$

and take a trivialization of $\mathscr{B}_n$. Then over a trivializing open subset U, $\mathscr{B}_n|_U \cong U \times (\mathbb{C}^{\oplus n} \bigoplus_{\alpha\in R_n} \mathbb{C}_\alpha)$. Take a Chevalley basis $\{x^U_\alpha, \alpha \in R_n; h_i, 1 \leq i \leq n\}$ for $\mathscr{B}_n|_U$ and define the Lie algebra structure by the following four relations, namely, Serre's relations on Chevalley basis (see [15], p147):

(a) $[h_i h_j] = 0, 1 \leq i, j \leq n$.
(b) $[h_i x^U_\alpha] = \langle\alpha, \alpha_i\rangle x^U_\alpha, 1 \leq i \leq n, \alpha \in R_n$.
(c) $[x^U_\alpha x^U_{-\alpha}] = h_\alpha$ is a $\mathbb{Z}$-linearly combination of $h_1, \cdots, h_n$.
(d) If α, β are independent roots, and $\beta - r\alpha, \cdots, \beta + q\alpha$ are the α-string through β, then $[x^U_\alpha x^U_\beta] = 0$ if $q = 0$, while $[x^U_\alpha x^U_\beta] = \pm(r+1)x^U_{\alpha+\beta}$ if $\alpha + \beta \in R_n$.

Note that $h_i, 1 \leq i \leq n$ are independent of any trivialization, so the relation (a) is always invariant under different trivializations. If $\mathscr{B}_n|_V \cong V \times (\mathbb{C}^{\oplus n} \bigoplus_{\alpha\in R_n})$ is another trivialization, and f^{UV}_α is the transition function for the line bundle $\mathcal{O}(\alpha)(\alpha \in R_n)$, that is, $x^U_\alpha = f^{UV}_\alpha x^V_\alpha$, then the relation (b) is

$$[h_i(f^{UV}_\alpha x^V_\alpha)] = \langle\alpha, \alpha_i\rangle f^{UV}_\alpha x^V_\alpha,$$

that is,

$$[h_i x^V_\alpha] = \langle\alpha, \alpha_i\rangle x^V_\alpha.$$

So (b) is also invariant. (c) is also invariant since $(f^{UV}_\alpha)^{-1}$ is the transition function for $\mathcal{O}(-\alpha)(\alpha \in R_n)$. Finally, (d) is invariant since $f^{UV}_\alpha f^{UV}_\beta$ is the transition function for $\mathcal{O}(\alpha + \beta)(\alpha, \beta \in R_n)$.

Therefore, the Lie algebra structure is compatible with the trivialization. Hence it is well-defined.

When the surface S contains Σ as an anti-canonical curve, restricting the above bundle to this anti-canonical curve Σ, we obtain a Lie algebra bundle of B_n-type over Σ, which determines uniquely a flat B_n bundle over Σ. On the other hand, when $x_1 = 0$, we can identify these two line bundles $\mathcal{O}_\Sigma(l_1)$ and $\mathcal{O}_\Sigma(f - l_1)$ when restricting them to Σ. Recall the spinor bundles S^+_{n+1} and S^-_{n+1} of D_{n+1} are defined as follows[19][20] (here we omit the subscription $n+1$ for brevity)

$$\begin{aligned} \mathcal{S}^+ &= \bigoplus_{D^2 = D\cdot K = -1, D\cdot f = 1} \mathcal{O}(D) \text{ and} \\ \mathcal{S}^- &= \bigoplus_{T^2 = -2, T\cdot K = 0, T\cdot f = 1} \mathcal{O}(T). \end{aligned}$$

The identification of $\mathcal{O}_\Sigma(l_1) \cong \mathcal{O}_\Sigma(f - l_1)$ induces an identification of these two spinor bundles S^+ and S^-, which is given by (of course, when restricted to Σ)

$$S^+ \otimes \mathcal{O}(-l_1) \cong S^-.$$

From representation theory, we know this determines a flat B_n bundle over Σ.

Conversely, if $\mathcal{S}^+|_\Sigma \cong \mathcal{S}^-|_\Sigma$, then we must have $x_1 = 0$ (up to renumbering). For example, we consider the $n = 2$ case. Note that

$$\begin{aligned} \mathcal{S}^+|_\Sigma \quad \otimes \quad & \mathcal{O}(-(0)) = \mathcal{O} \oplus \mathcal{O}((-x_1 - x_2) - (0)) \oplus \mathcal{O}((-x_1 - x_3) - (0)) \\ & \oplus \mathcal{O}((-x_2 - x_3) - (0)), \\ \mathcal{S}^-|_\Sigma \quad = \quad & \mathcal{O}((0) - (x_1)) \oplus \mathcal{O}((0) - (x_2)) \oplus \mathcal{O}((0) - (x_3)) \\ & \oplus \mathcal{O}(3(0) - (x_1) - (x_2) - (x_3)). \end{aligned}$$

Where for a point $x \in \Sigma$, (x) means the divisor of degree one, and $\mathcal{O}((x))$ means the line bundle determined by this divisor. Thus, $\mathcal{S}^+_\Sigma \otimes \mathcal{O}(-(0)) = \mathcal{S}^-_\Sigma$ implies that $x_1 = 0$ (up to renumbering). The general case follows from similar arguments.

2.2. **The C_n bundles.** We take $G = C_n \subset G' = A_{2n-1}$, where $C_n = Sp(n)$ and $A_{2n-1} = SU(2n)$. They satisfy the relation $G = (G')^{Out(G')}$.

Let $S = Z_{2n}$ be a rational surface with an A_{2n-1}-configuration (see [20] or Figure 3) which contains Σ as a smooth anti-canonical curve. Recall [20] that Z_{2n} is a (successive) blow-up of $\mathbb{F}_1$ at $2n$ points $x_1, \cdots, x_{2n}$ on Σ, with corresponding exceptional classes $l_1, \cdots, l_{2n}$. Let f be the class of fibers in $\mathbb{F}_1$, and s be the section such that $0 = s \cap \Sigma$ is the identity element of Σ. The Picard group of Z_{2n} is $H^2(Z_{2n}, \mathbb{Z})$, which is a lattice with basis $s, f, l_1, \cdots, l_{2n}$. The canonical line bundle $K = -(2s + 3f - \sum_{i=1}^{2n} l_i)$.

Recall

$$P_{2n-1} := \{x \in H^2(Z_{2n}, \mathbb{Z}) \mid x \cdot K = x \cdot f = x \cdot s = 0\}$$

is a root lattice of A_{2n-1} type. And we can take a simple root system of A_{2n-1} as

$$\Delta(A_{2n-1}) = \{\alpha_i = l_i - l_{i+1} \mid 1 \leq i \leq 2n - 1\}.$$

Note that [20] we have used the convention that $\sum_{i=1}^{2n} x_i = 0$.

Let ρ be the generator of $Out(G') \cong \mathbb{Z}_2$, such that $\rho(\alpha_i) = \alpha_{2n-i}$ for $i = 1, \cdots, 2n - 1$.

When the above simple root system is chosen, the pair (S, Σ) determines a homomorphism $u \in Hom(\Lambda(G'), \Sigma)$ which is given by the restriction map

$$u(\alpha) = \mathcal{O}(\alpha)|_\Sigma.$$

Lemma 17. *Let $u \in Hom(\Lambda(G'), \Sigma)$ be an element corresponding to a pair (S, Σ), where S is a surface with an A_{2n-1}-configuration. Then $\rho \cdot u = u$ if and only if $n(x_i + x_{2n+1-i}) = 0$ for $i = 1, \cdots, n$.*

Proof. Since u is the restriction map: $\alpha_i \mapsto \mathcal{O}(\alpha_i)|_\Sigma$, $u(\alpha_i) = \mathcal{O}(l_i - l_{i+1})|_\Sigma = x_i - x_{i+1}$ for $i = 1, \cdots, 2n-1$. Hence $\rho \cdot u = u \Leftrightarrow u(\alpha_i) = u(\alpha_{2n-i}) \Leftrightarrow x_i - x_{i+1} = x_{2n-i} - x_{2n-i+1} \Leftrightarrow n(x_i + x_{2n-i+1}) = 0$ since $\sum_{i=1}^{2n} x_i = 0$. □

As in [20], we denote $\mathcal{S}(\Sigma, G')$ the moduli space of $G' = A_{2n-1}$-surfaces with a fixed anti-canonical curve Σ, and $\overline{\mathcal{S}(\Sigma, G')}$ the natural compactification by including all rational surfaces with A_{2n-1}-configurations. From [20] we know that there is an isomorphism $\phi : \overline{\mathcal{S}(\Sigma, G')} \xrightarrow{\sim} \mathcal{M}^{G'}_\Sigma$.

Corollary 18. *For $u \in \mathcal{M}_\Sigma^G \hookrightarrow (\mathcal{M}_\Sigma^{G'})^{Out(G')}$, $\phi^{-1}(u) \in \overline{\mathcal{S}(\Sigma, G')}$ represents a class of surfaces $Z_{2n}(x_1, \cdots, x_{2n})$ with $x_i + x_{2n+1-i} = 0$ for $i = 1, \cdots, n$, and such a surface corresponds to a boundary point in the moduli space, that is, $\phi^{-1}(u) \in \overline{\mathcal{S}(\Sigma, G')} \backslash \mathcal{S}(\Sigma, G')$.*

Proof. By Lemma 17, $u \in (\mathcal{M}_\Sigma^{G'})^{Out(G')}$ if and only if $n(x_i + x_{2n+1-i}) = 0$ for $i = 1, \cdots, n$. There are n^2 connected components corresponding to n^2 points of order n on Σ. Since $\mathcal{M}_\Sigma^G$ is the component containing the trivial G' bundle, we have $x_i + x_{2n+1-i} = 0$ for $i = 1, \cdots, n$. Recall (§4, [20]) that $Z_{2n}(x_1, \cdots, x_{2n}) \in \mathcal{S}(\Sigma, G')$ if and only if $0, x_1, \cdots, x_{2n}$ are in general position, which implies in particular $x_i \neq -x_{2n+1-i}$. Hence $\phi^{-1}(u)$ corresponds to a boundary point. $\square$

Denote $S = Z'_{2n}(\pm x_1, \cdots, \pm x_n)$ the blow-up of $\mathbb{F}_1$ at n pairs of points $(x_1, -x_1)$, $\cdots$, $(x_n, -x_n)$ on Σ, with n pairs of corresponding exceptional divisors (l_1, l_1^-), $\cdots$, (l_n, l_n^-), where l_i (resp. l_i^-) is the exceptional divisor corresponding to the blowing up at x_i (resp. $-x_i$). Similar to the other cases, we give the following definitions.

Definition 19. A C_n*-exceptional system* on S is an n-tuple of pairs

$$((e_1, e_1^-), \cdots, (e_n, e_n^-))$$

where $(e_i, e_i^-) = (l_{\sigma(i)}, l_{\sigma(i)}^-)$ or $(l_{\sigma(i)}^-, l_{\sigma(i)})$, $i = 1, \cdots, n$, with σ is a permutation of $1, \cdots, n$. A C_n*-configuration* on S is a C_n-exceptional system $\zeta_{C_n} = ((e_1, e_1^-), \cdots, (e_n, e_n^-))$ such that we can blow down successively $e_1^-, \cdots, e_n^-, e_n, \cdots, e_1$ such that the resulting surface is $\mathbb{F}_1$ (see Figure 4).

We say that $x_1, x_2, \cdots, x_n \in \Sigma \subset \mathbb{F}_1$ are n points *in general position*, if they satisfy

(i) they are distinct points, and

(ii) for any i, j, $x_i + x_j \neq 0$.

Equivalently, $x_1, x_2, \cdots, x_n \in \Sigma \subset \mathbb{F}_1$ are in general position if and only if any C_n-exceptional system on $S = Z'_{2n}(\pm x_1, \cdots, \pm x_n)$ consists of smooth exceptional curves. Such a surface is called a C_n*-surface*. Thus a C_n-surface must have a C_n-configuration.

Lemma 20. *(i) Let S be a surface with a C_n-configuration. Then the Weyl group $W(C_n)$ acts on all C_n-exceptional systems on S simply transitively.*

(ii) Let S be a C_n-surface. Then the Weyl group $W(C_n)$ acts on all C_n- configurations on S simply transitively.

Proof. It suffices to prove (i). According to Remark 5, the Weyl group $W(C_n)$ acts as the group generated by permutations of the n pairs $\{(l_i, l_i^-) \mid i = 1, \cdots, n\}$ and interchanging of l_i and l_i^- for each i. From this, we see that $W(C_n)$ acts on all G-configurations simply transitively. $\square$

Denote $\mathcal{S}(\Sigma, C_n)$ the moduli space of pairs (Z'_{2n}, Σ), where Z'_{2n} is a C_n-surface, that is, the blow-up of $\mathbb{F}_1$ at $2n$ points $\pm x_1, \cdots, \pm x_n$ such that $x_1, \cdots, x_n$ are in general position. Denote $\mathcal{M}_\Sigma^{C_n}$ the moduli space of flat C_n bundles over Σ. By Corollary 18 we have the following identification.

Proposition 21. *(i) $\mathcal{S}(\Sigma, C_n)$ is embedded into $\mathcal{M}_\Sigma^{C_n}$ as an open dense subset.*

(ii) Moreover, this embedding can be extended naturally to an isomorphism

$$\overline{\mathcal{S}(\Sigma, C_n)} \cong \mathcal{M}_\Sigma^{C_n},$$

by including all rational surfaces with C_n-configurations.

Proof. The proof is basically the same as that in B_n case. We only need to replace the corresponding parts by the following two things. Firstly, according to Section 2, we can take a simple root system as

$$\Delta(C_n) = \{\beta_k = \varepsilon_k - \varepsilon_{k+1},\ 1 \leq k \leq n-1,\ \beta_n = 2\varepsilon_n\},$$

where $\varepsilon_k = l_k - l_k^-,\ 1 \leq k \leq n.$

Secondly, the restriction map gives us the following system of linear equations:

$$\begin{cases} 4x_n = p_n, \\ 2(x_k - x_{k+1}) = p_k,\ k = 1, \cdots, n-1. \end{cases}$$

The solution of this system exists uniquely. □

Remark 22. As in B_n case (Remark 16), the above proposition is also similar to extending the Torelli theorem to a certain boundary component.

Remark 23. Obviously, this description in Proposition 21 coincides with the well-known description of flat C_n bundles over elliptic curves [12]. A flat $C_n = Sp(n)$ bundle over Σ corresponds to n pairs (unordered) of points $(x_i, -x_i), i = 1, \cdots, n$ on Σ, uniquely up to isomorphism. And one pair $(x_i, -x_i)$ will determine exactly one point on $\mathbb{CP}^1$, since the rational map determined by the linear system $|2(0)|$ induces a double covering from Σ onto $\mathbb{CP}^1$. So the moduli space of flat C_n bundles over Σ is just isomorphic to $S^n(\mathbb{CP}^1) = \mathbb{CP}^n$, the ordinary projective n space.

As in B_n case, we construct a Lie algebra bundle of C_n type over Z'_{2n}:

$$\mathscr{C}_n = \mathcal{O}^{\bigoplus n} \bigoplus_{D \in R(C_n)} \mathcal{O}(D),$$

where $R(C_n)$ is the root system of C_n according to Section 2:

$$R(C_n) = \{\pm 2(l_i - l_i^-), \pm((l_i - l_i^-) \pm (l_j - l_j^-)) \mid i \neq j, 1 \leq i, j \leq n\}.$$

Recall [20] the first fundamental representation bundle of $\mathscr{A}_{2n-1}$ is

$$\mathcal{V}_{2n-1} = \bigoplus_{i=1}^{2n} \mathcal{O}(l_i).$$

The condition that $x_i + x_{2n+1-i} = 0, 1 \leq i \leq n$ is equivalent to an identification of the following two fundamental representation bundles $\wedge^i(\mathcal{V}_{2n-1})$ and $\wedge^{2n-i}(\mathcal{V}_{2n-1})$ with $i = 1, \cdots, n-1$, which is given by (of course, when restricted to Σ)

$$(\wedge^i(\mathcal{V}_{2n-1}))^* \otimes det(\mathcal{V}_{2n-1}) \cong \wedge^{2n-i}(\mathcal{V}_{2n-1}).$$

Note that when restricted to Σ, the line bundle $det(\mathcal{V}_{2n-1}) = \mathcal{O}(l_1 + \cdots l_{2n})$ is isomorphic to $\mathcal{O}(nf)|_\Sigma = \mathcal{O}_\Sigma(2n(0))$, by our assumption that $\sum x_i = 0$. This identification determines uniquely a flat C_n bundle over Σ.

2.3. **The G_2 bundles.** For $G = G_2$, we take $G' = D_4 = Spin(8)$ such that $G = (G')^{Out(G')}$.

Let $S = Y_4$ be a rational surface with a D_4-configuration [20] which contains Σ as a smooth anti-canonical curve. Recall ([20] or Figure 5) that Y_4 is a (successive) blow-up of $\mathbb{F}_1$ at 4 points $x_1, \cdots, x_4$ on Σ, with corresponding exceptional classes $l_1, \cdots, l_4$. Let f be the class of fibers in $\mathbb{F}_1$, and s be the section such that $0 = s \cap \Sigma$

is the identity element of Σ. The Picard group of Y_4 is $H^2(Y_4,\mathbb{Z})$, which is a lattice with basis $s,f,l_1,\cdots,l_4$. The canonical line bundle $K=-(2s+3f-\sum_{i=1}^{4} l_i)$.

Recall

$$P_4 := \{x\in H^2(Y_4,\mathbb{Z}) \mid x\cdot K = x\cdot f = 0\}$$

is a root lattice of D_4-type. And we can take a simple root system of D_4 as

$$\Delta(D_4)=\{\alpha_1=l_1-l_2,\alpha_2=f-l_1-l_2,\alpha_3=l_2-l_3,\alpha_4=l_3-l_4\}.$$

Let $\rho\in Out(G')\cong S_3$ (the permutation group of 3 letters) be the triality automorphism of order 3, such that $\rho(\alpha_1)=\alpha_2$, $\rho(\alpha_2)=\alpha_4$, $\rho(\alpha_4)=\alpha_1$, and $\rho(\alpha_3)=\alpha_3$.

When the above simple root system is chosen, the pair (S,Σ) determines a homomorphism $u\in Hom(\Lambda(G'),\Sigma)$ which is given by the restriction map

$$u(\alpha)=\mathcal{O}(\alpha)|_{\Sigma}.$$

Lemma 24. *Let $u\in Hom(\Lambda(G'),\Sigma)$ correspond to the pair (S,Σ), where S is a surface with a D_4-configuration. Then $\rho\cdot u=u$ if and only if $2x_1=0$ and $x_1+x_4=x_2+x_3$.*

Proof. Since u is the restriction map: $\alpha_i\mapsto\mathcal{O}(\alpha_i)|_{\Sigma}$, $u(\alpha_1)=\mathcal{O}(l_1-l_2)|_{\Sigma}=x_1-x_2$, $u(\alpha_2)=-x_1-x_2$, $u(\alpha_4)=x_3-x_4$, and $u(\alpha_3)=x_2-x_3$. Hence $\rho\cdot u=u$ $\Leftrightarrow u(\alpha_1)=u(\alpha_2)=u(\alpha_4)\Leftrightarrow x_1-x_2=-x_1-x_2=x_3-x_4\Leftrightarrow 2x_1=0$ and $x_1+x_4=x_2+x_3$. $\square$

Denote $\mathcal{S}(\Sigma,G')$ the moduli space of $G'=D_4$-surfaces with a fixed anti-canonical curve Σ, and $\overline{\mathcal{S}(\Sigma,G')}$ the natural compactification by including all rational surfaces with D_4-configurations. From [20] we know that $\overline{\mathcal{S}(\Sigma,G')}\xrightarrow{\sim}\mathcal{M}_{\Sigma}^{G'}$. Let ϕ be the isomorphism.

Corollary 25. *For $u\in\mathcal{M}_{\Sigma}^{G}\hookrightarrow(\mathcal{M}_{\Sigma}^{G'})^{Out(G')}$, $\phi^{-1}(u)\in\overline{\mathcal{S}(\Sigma,G')}$ represents a class of surfaces $Y_4(x_1,\cdots,x_4)$ with $x_1=0$ and $x_4=x_2+x_3$, and such a surface corresponds to a boundary point in the moduli space, that is, $\phi^{-1}(u)\in\overline{\mathcal{S}(\Sigma,G')}\backslash\mathcal{S}(\Sigma,G')$.*

Proof. By Lemma 24, $u\in(\mathcal{M}_{\Sigma}^{G'})^{Out(G')}$ if and only if $2x_1=0$ and $x_1+x_4=x_2+x_3$. There are 4 connected components corresponding to 4 points of order 2 on Σ. Since $\mathcal{M}_{\Sigma}^{G}$ is the component containing the trivial G' bundle, we have $x_1=0$ and $x_4=x_2+x_3$. Recall that $Y_4(x_1,\cdots,x_4)\in\mathcal{S}(\Sigma,G')$ if and only if $0,x_1,\cdots,x_4$ are in general position, which implies in particular $x_1\neq0$. Hence $\phi^{-1}(u)$ corresponds to a boundary point. $\square$

Denote $S=Y_4'(x_1,\cdots,x_4)$ the blow-up of $\mathbb{F}_1$ at 4 points $x_1,\cdots,x_4$ on Σ, with $x_1=0$ and $x_4=x_2+x_3$. Let $l_1,\cdots,l_4$ be the corresponding exceptional classes. We give the following definition.

Definition 26. A G_2-*exceptional system* on S is an ordered triple (e_1,e_2,e_3,e_4) of exceptional divisors such that $e_i\cdot e_j=0=e_i\cdot f, i\neq j$ and $y_1=0$, $y_4=y_2+y_3$ where $y_i=e_i\cdot\Sigma$. A G_2-*configuration* on S is a G_2-exceptional system $\zeta_{G_2}=(e_1,e_2,e_3,e_4)$ such that we can consider S as a blow-up of $\mathbb{F}_1$ at these 4 points $y_1=0,y_2,y_3,y_4$ on Σ, that is $S=Y_4'(y_1=0,y_2,y_3,y_4)$, with corresponding exceptional divisors e_1,e_2,e_3,e_4. When S has a G_2-configuration (of course $\Sigma\in|-K_S|$), we call S a *(rational) surface with a G_2-configuration.* For $S=Y_4'(x_1,\cdots,x_4)$ with $x_1=0$ and

$x_4 = x_2 + x_3$, when $x_1, \pm x_2, \pm x_3, \pm x_4$ are distinct points on Σ, any G_2-exceptional system on S consists of exceptional curves. Such a surface is called a *G_2-surface*. So a G_2-surface must have a G_2-configuration. These four points $x_1, x_2, x_3, x_4 \in \Sigma$ are said to be *in general position*.

A G_2-configuration is illustrated in Figure 6.

Lemma 27. *(i) Let $S = Y_4'(x_1, \cdots, x_4)$ with $x_1 = 0$ and $x_4 = x_2 + x_3$ be a surface with a G_2-configuration. Then the Weyl group $W(G_2)$ acts on all G_2-exceptional systems on S simply transitively.*

(ii) Let S be a G_2-surface. Then the Weyl group $W(G_2)$ acts on all G_2-configurations on S simply transitively.

Proof. It suffices to prove (i). By an explicit computation, there are 12 G_2-configurations: (l_1, l_2, l_3, l_4), $(f - l_1, f - l_2, f - l_3, f - l_4)$, $(f - l_1, f - l_2, l_4, l_3)$, $(f - l_1, l_4, f - l_2, l_3)$, and so on. The rule is keeping the relation $x_2 + x_3 = x_4$ fixed. The Weyl group $W(G_2)$ is the automorphism group of the sub-root system A_2 with simple roots $\{3(l_2 - l_3), 3(l_3 - (f - l_4))\}$, so $W(G_2) \cong \mathbb{Z}_2 \rtimes W(A_2) = \mathbb{Z}_2 \rtimes S_3$. We can also consider $W(G_2)$ as the subgroup of $W(D_4)$ generated by two elements $S_{\alpha_1} S_{\alpha_2} S_{\alpha_4}$ and S_{α_3}, where S_α means the reflection with respect to a root α of D_4, according to Remark 5. Thus we can directly check that $W(G_2)$ acts on all G_2-exceptional systems simply transitively. $\square$

Proposition 28. *Let $\mathcal{S}(\Sigma, G_2)$ be the moduli space of pairs (Y_4', Σ) where Y_4' is a G_2-surface, and $\mathcal{M}_\Sigma^{G_2}$ be the moduli space of flat G_2 bundles over Σ. Then we have*

(i) $\mathcal{S}(\Sigma, G_2)$ is embedded into $\mathcal{M}_\Sigma^{G_2}$ as an open dense subset.

(ii) Moreover, this embedding can be extended naturally to an isomorphism

$$\overline{\mathcal{S}(\Sigma, G_2)} \cong \mathcal{M}_\Sigma^{G_2},$$

by including all rational surfaces with G_2-configurations.

Proof. We just note that only the following two things are different from their counterparts of the proofs in B_n, C_n cases.

(i) Take a simple root system of G_2 as (Remark 4)

$$\Delta(G_2) = \{\beta_1 = f - 2l_2 + l_3 - l_4,\ \beta_2 = 3(l_2 - l_3)\}.$$

(ii) Then the restriction to Σ gives us the following system of linear equations:

$$\begin{cases} 3x_2 = -p_1, \\ 3(x_2 - x_3) = p_2. \end{cases} \qquad \square$$

As before, we construct a Lie algebra bundle of G_2-type over $S = Y_4'$. For brevity, denote $\varepsilon_1 = l_2,\ \varepsilon_2 = l_3,$ and $\varepsilon_3 = f - l_4$. Then

$$\mathscr{G}_2 = \mathcal{O}^{\bigoplus 2} \bigoplus_{D \in R(G_2)} \mathcal{O}(D),$$

where $R(G_2)$ is the root system of G_2:

$$R(G_2) = \{\pm 3(\varepsilon_i - \varepsilon_j), \pm(2\varepsilon_i - \varepsilon_j - \varepsilon_k) \mid i \neq j \neq k, 1 \leq i, j, k \leq 3\},$$

according to Remark 4.

Recall [19] the 3 fundamental representation bundles of rank 8 of D_4 are defined as:

$$\begin{cases} \mathscr{W}_4 = \bigoplus_{C^2=C\cdot K=-1, C\cdot f=0} \mathcal{O}(C), \\ \mathcal{S}_4^+ = \bigoplus_{D^2=D\cdot K=-1, D\cdot f=1} \mathcal{O}(D), \\ \mathcal{S}_4^- = \bigoplus_{T^2=-2, T\cdot K=0, T\cdot f=1} \mathcal{O}(T). \end{cases}$$

These conditions $x_1 = 0,\ x_4 = x_2 + x_3$ enable us to identify S_4^+, S_4^- and $\mathscr{W}_4$ when restricted to Σ, by

$$S_4^+ \otimes \mathcal{O}(-l_1) \cong S_4^- \text{ and } S_4^+ \cong \mathscr{W}_4 \otimes \mathcal{O}(s).$$

And these identifications determine uniquely a flat G_2 bundle over Σ. Conversely, the identification of these three bundles restricted to Σ implies the conditions $x_1 = 0$ and $x_4 = x_2 + x_3$ (up to renumbering). Note that

$$\begin{aligned} \mathscr{W}_4|_\Sigma &= \bigoplus \mathcal{O}_\Sigma(l_i) \bigoplus \mathcal{O}_\Sigma(f - l_i) = \bigoplus \mathcal{O}((x_i)) \bigoplus \mathcal{O}((-x_i)), \\ \mathcal{S}_4^-|_\Sigma &= \bigoplus_i \mathcal{O}((0) - (x_i)) \bigoplus_j \mathcal{O}(3(0) - \sum_{i\neq j}(x_i)), \text{ and} \\ \mathcal{S}_4^+|_\Sigma &= \mathcal{O}((0)) \bigoplus_{i\neq j} \mathcal{O}((-x_i - x_j)) \bigoplus \mathcal{O}((-\sum x_i)). \end{aligned}$$

So $\mathscr{W}_4|_\Sigma = \mathcal{S}_4^-$ implies $x_1 = 0$, and $\mathscr{W}_4|_\Sigma = \mathcal{S}_4^+$ implies $x_4 = x_2 + x_3$.

2.4. **The F_4 bundles.** First we recall some fundamental facts on E_6 root systems and cubic surfaces, which are of independent interest.

2.4.1. *The root system of E_6, revisited.* The relation between the root system of E_6-type and smooth cubic surfaces in $\mathbb{CP}^3$ has been studied for a very long time [14][6][24]. There are 27 *lines* on such a cubic surface S (a curve on S is a line if and only if it is an exceptional curve). And every E_6-*exceptional system* on S is an ordered 6-tuples of lines $(e_1, \cdots, e_6)$ which are pairwise disjoint. The Weyl group $W(E_6)$ is the symmetry group of all E_6-exceptional systems, that is, $W(E_6)$ acts simply transitively on the set of all E_6-exceptional systems. Now we consider the unordered 6-tuple $L = \{e_1, \cdots, e_6\}$. There are 72 such 6-tuples. This corresponds to 36 *Schläfli's double-sixes* $\{L; L'\}$ [14]. In the following we consider a cubic surface S as the blow-up of $\mathbb{P}^2$ at 6 points $x_1, \cdots, x_6$ in general position, that is $S = X_6(x_1, \cdots, x_6)$, with corresponding exceptional curves $l_1, \cdots, l_6$. Fix a simple root system of E_6 as

$$\Delta(E_6) = \{\alpha_1, \cdots, \alpha_6\},$$

where $\alpha_1 = l_1 - l_2$, $\alpha_2 = l_2 - l_3$, $\alpha_3 = h - l_1 - l_2 - l_3$, and $\alpha_i = l_{i-1} - l_i$, for $i = 4, 5, 6$ [20].

Lemma 29. *One double-six $\{L; L'\}$ corresponds to exactly one positive root of E_6.*

Proof. First take $L_0 = \{l_1, \cdots, l_6\}$, then $L_0' = \{l_1', \cdots, l_6'\} = s_{\alpha_0}(L_0)$ where $\alpha_0 = 2h - \sum l_i$ is a positive root and $l_i' = s_{\alpha_0}(l_i) = 2h - \sum_{j\neq i} l_j$. $\{L_0; L_0'\}$ forms a double-six and $\alpha_0(\succ 0)$ is uniquely determined by $\{L_0; L_0'\}$, since $W(E_6)$ acts simply and transitively. If $L = g(L_0)$ with $g \in W(E_6)$, then $\{g(L_0); g(L_0')\}$ is also a double-six. Let $g(L_0') = S_\alpha(g(L_0))$, then $L_0' = (g^{-1}S_\alpha g)(L_0)$. So $g^{-1}S_\alpha g = S_{\alpha_0}$. Then $S_\alpha = gS_{\alpha_0}g^{-1} = S_{g(\alpha_0)}$. This implies $\alpha = \pm g(\alpha_0)$. Take $\alpha \succ 0$. Now if

$\alpha = \alpha_0$, then by a result in page 44 of [16], $g \in S_6$, that is, g is a permutation of the six lines l_i's. Thus $\{L; L'\}$ and $\{L_0; L_0'\}$ are the same one. □

Remark 30. Let ρ be an outer automorphism of E_6 of order 2, such that $\rho(\alpha_1) = \alpha_6, \rho(\alpha_2) = \alpha_5$ and ρ fixes other simple roots. Consider F_4 as the fixed part of E_6 by ρ. Then the coroot lattice $\Lambda_c(F_4)$ of F_4 is

$$\begin{aligned}\Lambda_c(F_4) &= \Lambda_c(E_6)^\rho \\ &= \Lambda(E_6)^\rho \\ &= \{ah + \sum a_i l_i \mid a_1 + a_6 = a_2 + a_5 = a_3 + a_4 = -a\} \\ &= \mathbb{Z}\langle h - l_1 - l_2 - l_3, l_1 - l_6, l_2 - l_5, l_3 - l_4\rangle \\ &= \Lambda(D_4).\end{aligned}$$

And the Weyl group of F_4 is

$$\begin{aligned}W(F_4) &= \{w \in W(E_6) \mid w \text{ preserves } \Lambda_c(F_4) = \Lambda(D_4)\} \\ &= Aut(\Lambda(D_4)) \\ &= S_3 \rtimes W(D_4).\end{aligned}$$

Remark 31. If 3 lines e_1, e_2, e_3 pairwise intersect, we say that they form a *triangle*. Denote by $\Delta = \{e_1, e_2, e_3\}$ a (unordered) triangle, and by $\overrightarrow{\Delta} = (e_1, e_2, e_3)$ an ordered triangle. Every line belongs to 5 triangles, so there are $27 \cdot 5/3 = 45$ triangles. And if $\{e_1, e_2, e_3\}$ is a triangle, then $-K = e_1 + e_2 + e_3$. $W(E_6)$ acts on all these 45 triangles transitively, and $W(F_4)$ is the isotropy subgroup of the triangle $\Delta_0 = \{h - l_1 - l_6, h - l_2 - l_5, h - l_3 - l_4\}$. Moreover $W(D_4)$ is the isotropy subgroup of the ordered triangle $\overrightarrow{\Delta}_0 = (h - l_1 - l_6, h - l_2 - l_5, h - l_3 - l_4)$. The reason is the following:

Let $\Delta = \{e_1, e_2, e_3\}$ and $\Delta' = \{f_1, f_2, f_3\}$ be any two triangles. Since $K^2 = 3$, the position of these two triangles must be one of the following two cases. (1) They have a common edge and other edges don't intersect. (2) Each edge of Δ intersects with exactly one edge of Δ'. So we just check two special triangles in above cases. what remains to do is a direct checking.

From above we can easily write down the 45 (left or right) cosets of $W(F_4)$ in $W(E_6)$.

2.4.2. *F_4 bundles and rational surfaces.* For $G = F_4$ we take $G' = E_6$, such that $F_4 = (E_6)^{Out(E_6)}$.

Let $S = X_6(x_1, \cdots, x_6)$ be a surface with an E_6-configuration (Figure 7), that is, S is a blow-up of $\mathbb{P}^2$ at 6 points $x_1, \cdots, x_6 \in \Sigma$, where $\Sigma \in |-K_S|$. Take the simple root system $\Delta(E_6)$ and $\rho \in Out(E_6)$ just as in Section 2.4.1.

Once a simple root system is fixed, the restriction from S to Σ induces a homomorphism $u \in Hom(\Lambda(E_6), \Sigma)$.

Lemma 32. *Let $u \in Hom(\Lambda(E_6), \Sigma)$ be an element corresponding to a pair (S, Σ), where S is a surface with an E_6-configuration. Then $\rho \cdot u = u$ if and only if $x_1 + x_6 = x_2 + x_5 = x_3 + x_4$.*

Proof. Since u is induced by the restriction to Σ, $u(\alpha_1) = \mathcal{O}(l_1 - l_2)|_\Sigma = x_1 - x_2$, $u(\alpha_2) = x_2 - x_3$, $u(\alpha_5) = x_4 - x_5$, $u(\alpha_6) = x_5 - x_6$. Therefore $\rho \cdot u = u \Leftrightarrow$

$u(\alpha_1) = u(\alpha_6), u(\alpha_2) = u(\alpha_5) \Leftrightarrow x_1 + x_6 = x_2 + x_5 = x_3 + x_4$. □

Denote $\mathcal{S}(\Sigma, E_6)$ the moduli space of $G' = E_6$-surfaces [20] with a fixed anti-canonical curve Σ, and $\overline{\mathcal{S}(\Sigma, E_6)}$ the natural compactification by including all rational surfaces with E_6-configurations. From [20] we know that there is an isomorphism $\phi : \overline{\mathcal{S}(\Sigma, E_6)} \xrightarrow{\sim} \mathcal{M}_{\Sigma}^{E_6}$. Thus we have

Corollary 33. *For $u \in \mathcal{M}_{\Sigma}^{F_4} \subset (\mathcal{M}_{\Sigma}^{E_6})^{Out(E_6)}$, $\phi^{-1}(u) \in \overline{\mathcal{S}(\Sigma, E_6)}$ represents a class of surfaces $X_6(x_1, \cdots, x_6)$ with $x_1 + x_6 = x_2 + x_5 = x_3 + x_4$.* □

Denote $S = X_6'(x_1, \cdots, x_6)$ the blow-up of $\mathbb{P}^2$ at 6 points $x_1, \cdots, x_6$ on Σ which satisfies the condition $x_1 + x_6 = x_2 + x_5 = x_3 + x_4$, with corresponding exceptional classes $l_1, \cdots, l_6$. The condition $x_1 + x_6 = x_2 + x_5 = x_3 + x_4 := p$ implies that the three lines L_{16}, L_{25} and L_{34} in $\mathbb{P}^2$ intersect at one points $-p \in \Sigma$, where L_{ij} means the line in $\mathbb{P}^2$ passing through these two points x_i and x_j. So after blowing up $\mathbb{P}^2$ at $x_i \in \Sigma, 1 \leq i \leq 6$, the three (-1) curves $h - l_1 - l_6, h - l_2 - l_5$ and $h - l_3 - l_4$ intersect at one points $-p \in \Sigma$. So they form a special triangle (see Section 2.4.1). As before, we give the following definition.

Definition 34. An F_4-*exceptional system* on $S = X_6'$ is a 6-tuple $(e_1, \cdots, e_6)$ consisting of 6 exceptional divisors which are pairwise disjoint, such that $y_1 + y_6 = y_2 + y_5 = y_3 + y_4$, where $\mathcal{O}_{\Sigma}(y_i) = \mathcal{O}(e_i)|_{\Sigma}$. And an F_4-*configuration* $\zeta_{F_4} = (e_1, \cdots, e_6)$ just means an F_4-exceptional system on S such that we can consider S as a blow-up of $\mathbb{P}^2$ at 6 points $y_1, \cdots, y_6$ with corresponding exceptional divisors $e_1, \cdots, e_6$. For $S = X_6'(x_1, \cdots, x_6)$, when $x_1, \cdots, x_6$ are in general position, any F_4-exceptional system on S consists of exceptional curves. Such a surface is called an F_4-*surface.*

So an F_4-surface is automatically an E_6-surface (namely, a del Pezzo surface of degree 3). And any F_4-exceptional system on an F_4-surface is always an F_4-configuration. See Figure 8 for an F_4-configuration.

According to the discussions in Section 2.4.1, the Weyl group $W(F_4)$ is the automorphism group of the sub-root system of type D_4 with simple roots $\{l_1 - l_6, l_2 - l_5, l_3 - l_4, h - l_1 - l_2 - l_3\}$, and $W(F_4) \cong S_3 \rtimes W(D_4)$. Therefore we have

Lemma 35. *(i) Let $S = X_6'$ be a surface with an F_4-configuration. Then the Weyl group $W(F_4)$ acts on all F_4-exceptional systems on S simply transitively.*

(ii) Moreover, if S is an F_4-surface, then the Weyl group $W(F_4)$ acts on all F_4-configurations on S simply transitively. □

Proposition 36. *Let $\mathcal{S}(\Sigma, F_4)$ be the moduli space of pairs (X_6', Σ) where X_6' is an F_4-surface containing Σ as an anti-canonical curve, and $\mathcal{M}_{\Sigma}^{F_4}$ be the moduli space of flat F_4 bundles over Σ. Then we have*

(i) $\mathcal{S}(\Sigma, F_4)$ is embedded into $\mathcal{M}_{\Sigma}^{F_4}$ as an open dense subset.

(ii) Moreover, this embedding can be extended naturally to an isomorphism

$$\overline{\mathcal{S}(\Sigma, F_4)} \cong \mathcal{M}_{\Sigma}^{F_4},$$

by including all rational surfaces with F_4-configurations.

Proof. Firstly, we can take the simple root system of F_4 as

$$\Delta(F_4) = \{\beta_1, \beta_2, \beta_3, \beta_4\},$$

where $\beta_1 = l_1 - l_2 + l_5 - l_6$, $\beta_2 = l_2 - l_3 + l_4 - l_5$, $\beta_3 = 2(h - l_1 - l_2 - l_3)$, and $\beta_4 = 2(l_3 - l_4)$, according to Remark 4.

Secondly, the restriction to Σ induces the following system of linear equations:

$$\begin{cases} x_1 - x_2 + x_5 - x_6 = p_1, \\ x_2 - x_3 + x_4 - x_5 = p_2, \\ 2(-x_1 - x_2 - x_3) = p_3, \\ 2(x_3 - x_4) = p_4, \\ x_1 + x_6 = x_2 + x_5 = x_3 + x_4. \end{cases}$$

Since the determinant is non-zero, the result follows by the same argument as in B_n case. □

The Lie algebra bundle of type F_4 over X_6' can be constructed as (for brevity, we denote $\varepsilon_1 = l_2 - l_3 + l_4 - l_5$, $\varepsilon_2 = l_2 + l_3 - l_4 - l_5$, $\varepsilon_3 = 2h - 2l_1 - l_2 - l_3 - l_4 - l_5$, and $\varepsilon_4 = 2h - 2l_6 - l_2 - l_3 - l_4 - l_5$)

$$\mathscr{F}_4 = \mathcal{O}^{\bigoplus 4} \bigoplus_{D \in R(F_4)} \mathcal{O}(D),$$

where $R(F_4)$ is the root system of F_4:

$$R(F_4) = \{\pm\varepsilon_i,\ \pm(\varepsilon_i \pm \varepsilon_j),\ \pm\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4) \mid i \neq j\}.$$

Remark 37. The 27 lines determine the 27-dimensional fundamental representation of E_6. Restricted to Σ, they give us a representation bundle of rank 27 (of $\mathscr{F}_4$) over Σ. The weights associated to the 3 special lines $h - l_1 - l_6, h - l_2 - l_5, h - l_3 - l_4$ restrict to zero and these 3 weights add to zero before restriction (since $(h - l_1 - l_6) + (h - l_2 - l_5) + (h - l_3 - l_4) = -K$). The remaining 24 weights associated to other 24 lines restrict to the 24 short roots of $\mathscr{F}_4$. The 24 lines and a rank 2 bundle V determine the 26-dimensional irreducible fundamental representation U of $\mathscr{F}_4$. Here V is determined as follows. Since $\mathcal{O}_\Sigma(h - l_1 - l_6) = \mathcal{O}_\Sigma(h - l_2 - l_5) = \mathcal{O}_\Sigma(h - l_3 - l_4) = \mathcal{O}_\Sigma((-p))$, taking the trace, we have the following exact sequence:

$$0 \to ker(tr) \to \mathcal{O}_\Sigma((-p))^{\bigoplus 3} \to \mathcal{O}_\Sigma((-p)) \to 0.$$

Then we take $V = ker(tr)$.

For more details on the 26-dimensional fundamental representation of F_4, one can consult [1].

3. Conclusion

Let G be any simple, compact and simply connected Lie group. Then G is classified into the following 7 types according to its Lie algebra.

(1) A_n-type, $G = SU(n+1)$;
(2) B_n-type, $G = Spin(2n+1)$;
(3) C_n-type, $G = Sp(n)$;
(4) D_n-type, $G = Spin(2n)$;
(5) E_n-type, $n = 6, 7, 8$;
(6) F_4-type;
(7) G_2-type.

Among these, A_n, D_n and E_n are called of simply laced type, while B_n, C_n, F_4 and G_2 are called of non-simply laced type. And A_n, B_n, C_n, D_n are called classic Lie groups, while E_n, F_4 and G_2 are called exceptional Lie groups.

We summarize our results in [20] and this paper as follows. Let Σ be a fixed elliptic curve with identity $0 \in \Sigma$. Let G be any compact, simple and simply connected Lie groups, simply laced or not. Denote $\mathcal{S}(\Sigma, G)$ the moduli space of G-surfaces containing a fixed anti-canonical curve Σ. Denote $\mathcal{M}_\Sigma^G$ the moduli space of flat G bundles over Σ. Then we have

Theorem 38. *(i) We can construct Lie algebra Lie(G)-bundles over each G-surface.*

(ii) The restriction of these Lie algebra bundles to the anti-canonical curve Σ induces an embedding of $\mathcal{S}(\Sigma, G)$ into $\mathcal{M}_\Sigma^G$ as an open dense subset.

(iii) This embedding can be extended to an isomorphism from $\overline{\mathcal{S}(\Sigma, G)}$ onto $\mathcal{M}_\Sigma^G$, where $\overline{\mathcal{S}(\Sigma, G)}$ is a natural and explicit compactification of $\mathcal{S}(\Sigma, G)$, by including all rational surfaces with G-configurations. $\square$

Remark 39. (i) The result is known for $G = E_n$ case (see [7][8][10][12]).

(ii) We have mentioned in the beginning of § 1 that there is another reduction of the non-simply laced cases to simply laced cases. In fact, using this reduction, we will obtain the same result, just following the steps as above.

According to Looijenga's theorem [21][22], the moduli space $\mathcal{S}(\Sigma, G)$ is a weighted projective space. Thus the compactification $\overline{\mathcal{S}(\Sigma, G)}$ is a weighted projective space. Conversely, we believe that the above identification between $\mathcal{S}(\Sigma, G)$ and $\overline{\mathcal{S}(\Sigma, G)}$ will give us another proof for Looijenga's theorem. This is already done in E_n case by [7][8][10][12] and so on.

Acknowledgements. The second author would like to express his gratitude to Kwok-Wai Chan, Changzheng Li and Xiaowei Wang.

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