Mirror Symmetry of Fourier-Mukai transformation for Elliptic Calabi-Yau manifolds

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Abstract

Mirror symmetry conjecture identifies the complex geometry of a Calabi-Yau manifold with the symplectic geometry of its mirror Calabi-Yau manifold. Using the SYZ mirror transform, we argue that (i) the mirror of an elliptic Calabi-Yau manifold admits a twin Lagrangian fibration structure and (ii) the mirror of the Fourier-Mukai transform for dual elliptic fibrations is a symplectic Fourier-Mukai transform for dual twin Lagrangian fibrations, which is essentially an identity transformation in this case.

1 Introduction

Mirror symmetry conjecture says that for any Calabi-Yau manifold M near the large complex/symplectic structure limit, there is another Calabi-Yau manifold X, called the *mirror manifold*, such that the B-model superstring theory on M is equivalent to the A-model superstring theory on X, and vice versa. Mathematically speaking, it roughly says that the complex geometry of M is equivalent to the symplectic geometry of X, and vice versa. It is conjectured [22] that this duality can be realized as a Fourier type transformation along fibers of special Lagrangian fibrations on M and X, called the *SYZ mirror transformation* $\mathcal{F}^{SYZ}$.

Suppose M has an elliptic fibration structure,

$$p : M \to S.$$

then there is another manifold W with a *dual* elliptic fibration over S. Since any elliptic curve is isomorphic to its dual, we have actually $M \cong W$ provided that p has irreducible fibers. There is also a Fourier-Mukai transformation (abbrev. FM transform) $\mathcal{F}_{cx}^{FM}$ between the complex geometries of M and W. On the level of derived category of coherent sheaves, $\mathcal{F}_{cx}^{FM}$ is an equivalence of categories. On the level of cycles, this can be described as a *spectral cover construction* and it is a very powerful tool in the studies of holomorphic vector bundles over M.

In this paper we address the following two questions: (1) What is the SYZ transform of the elliptic fibration structure on M? (2) What is the SYZ transform of the Fourier-Mukai transformation $\mathcal{F}_{cx}^{FM}$?

The answer to the first question is a *twin Lagrangian fibration* structure on the mirror manifold X, coupled with a *superpotential*. To simplify the matter, we will ignore the superpotential in our present discussions. Similarly there is a twin Lagrangian fibration structure on the mirror manifold Y to W. We will explain several important properties of twin Lagrangian fibrations. In particular, we show that the twin Lagrangian fibration on Y is dual to the twin Lagrangian fibration on X. There is also an identification between X and Y, which is analogous to the identification between total spaces of dual elliptic fibrations M and W.

For the second question, the SYZ transform of $\mathcal{F}_{cx}^{FM}$ should be a symplectic Fourier-Mukai transformation $\mathcal{F}_{sym}^{FM}$ from X to Y. We will argue that this is actually the identity transformation! A naive explanation of this is because $\mathcal{F}_{sym}^{FM} = \mathcal{F}^{SYZ} \circ \mathcal{F}_{cx}^{FM} \circ \mathcal{F}^{SYZ}$ and each of the two $\mathcal{F}^{SYZ}$ transforms undo half of the complex FM transform.

The plan of the paper is as follow: In section 2 we review the SYZ mirror transformation and show that the mirror manifold to an elliptically fibered Calabi-Yau manifold has a twin Lagrangian fibration structure. In section 3 we review the FM transform in complex geometry in general and also for elliptic manifolds. In section 4 we first define the *symplectic* FM transform between Lagrangian cycles on X and Y. Then we define twin Lagrangian fibrations, give several examples of them and study their basic properties. In section 5 we show that the SYZ transformation of the complex FM transform between M and W is the symplectic FM transform between X and Y, which is actually the identity transformation.

2 Mirror Symmetry and SYZ Transformation

2.1 Geometry of Calabi-Yau manifolds

A real $2n$ dimensional Riemannian manifold M is a *Calabi-Yau manifold* (abbrev. CY manifold) if the holonomy group of its Levi-Civita connection is a subgroup of $SU(n)$. Equivalently, a CY manifold is a Kähler manifold with a parallel holomorphic volume form. A theorem [27] of the second author says that any compact Kähler manifold with trivial canonical line bundle admits such a structure.

The complex geometry of M includes the study of (i) the moduli space of complex structures on M, (ii) complex submanifolds, holomorphic vector bundles and Hermitian Yang-Mills metrics and (iii) the derived category $D^b(M)$ of coherent sheaves on M. The symplectic geometry of M includes the study of (i) the moduli space of (complexified) symplectic structures on M, (ii) Lagrangian submanifolds and their intersection theory and (iii) the Fukaya-Floer category $Fuk(M)$ of Lagrangian submanifolds in M. The complex geometry is more nonlinear in nature, whereas the symplectic geometry requires the inclusion of quantum corrections, in which contributions from holomorphic curves in M needed to be included.

2.2 Mirror symmetry conjectures

Roughly speaking, the mirror symmetry conjecture says that for mirror CY manifolds M and X, the complex geometry of M is equivalent to the symplectic geometry of X and vice versa.

$$\text{Complex Geometry}\left((M)\right) \quad \xleftrightarrow{\text{Mirror symmetry}} \quad \text{Symplectic Geometry}\left((X)\right).$$

This conjecture has far reaching consequences in many different parts of mathematics and physics. For instance, (i) Candelas et al [3] studied the identification between the moduli of complex structures on M with the moduli of complexified symplectic structures on X and derived an amazing formula which enumerative the number of rational curves of each degree in the quintic CY threefold. This mirror formula has been proved mathematically by Liu, Lian and the second author [19] and Givental [11] independently. (ii) A theorem of Donaldson [5] Uhlenbeck and the second author [24] related the existence of Hermitian Yang-Mills metrics to the stability of holomorphic vector bundles. Thomas and the second author [23] conjectured a mirror phenomenon to this for special Lagrangian submanifolds. (iii) Kontsevich's homological mirror conjecture [14] identifies $D^b(M)$ with $Fuk(X)$.

In this paper, we use the SYZ transform to study the mirror of an elliptic fibration structure on M and the FM transform associated to it.

2.3 SYZ transform

In order to explain the origin of this duality, Strominger, Yau and Zaslow [22] used physical reasonings to argue that (i) both M and X admit special Lagrangian tori fibrations with sections, which are fiberwise dual to each other. These are called SYZ fibrations. (ii) The equivalence between these two type of geometries is given by a geometric version of the Fourier-Mukai type transformation between M and X. We call this the SYZ transformation (see e.g. [16][17][18]).

To see this, note that the manifold M itself is the moduli space of certain complex cycles in M, namely points in M. Therefore M should also be the moduli space of certain special Lagrangian submanifolds with flat $U(1)$ bundles in X. These special Lagrangian submanifolds form a Lagrangian (tori) fibration on X since points form a fibration on M in a trivial way. By considering all those points in M which correspond to the same special Lagrangian torus in X but with different flat $U(1)$ connections, we know that M also admit a tori fibration which is naturally dual to the one on X. This is because the moduli space of flat $U(1)$ connections on a torus is naturally its dual torus.

$$\begin{array}{ccc} T & & T^* \\ \downarrow & & \downarrow \\ M & & X \\ \downarrow & & \downarrow \\ B & & B \end{array}$$

Dual Lagrangian fibrations

on mirror manifolds

Similarly the mirror of the complex cycle M itself is a special Lagrangian section for $X \to B$. This heuristic reasoning is only expected to hold true asymptotically near the large complex structure limit (LCSL) [22]. We can apply this SYZ transform to other coherent sheaves on M. For example, suppose L is a holomorphic line bundle on M, which restrict to a flat bundle on each fiber $T_b \subset M$ of the special Lagrangian fibration. Such a flat connection determines a point in the dual torus $T_b^* \subset X$. By varying $b \in B$, we obtain a Lagrangian section to $X \to B$, which is the mirror to L. This fiberwise Fourier transform forms the backbone of the *SYZ mirror transform.*

2.4 Mirror of elliptic fibrations

In this section we continue our reasonings to explain why the mirror manifold X to a CY manifold M with an elliptic fibration $p : M \to S$ admits another special Lagrangian fibration which is compatible with the original SYZ fibration on X. If the elliptic fibration on M has a section, then the corresponding special Lagrangian fibration on X also admit an appropriate section. Recall that the way we obtain the SYZ fibration $\pi : X \to B$ on X is by viewing the identity map $M \to M$ as a fibration on M by complex cycles and with a holomorphic section, both in trivial manners.

We consider the moduli space of coherent sheaves on M whose generic member has the same Hilbert polynomial as $\iota_* O_F$ where F is an elliptic fiber of $M \to S$ and $\iota : F \to M$ is the inclusion morphism. Geometrically speaking, this is a moduli space of elliptic curves in M together with holomorphic line bundles over them which are trivial topologically. This moduli space is nothing but the total space of the dual elliptic fibration $p' : W \to S$.

By the principle of mirror symmetry, we should view W also as the moduli space of certain Lagrangian cycles in the mirror manifold X. As $\dim_{\mathbb{C}} W = n$ and W consists of geometric cycles which foliate M, we can argue as before that X should have another Lagrangian fibration $p : X \to C$ such that W is also the moduli space of its Lagrangian fibers together with flat $U(1)$ connections over them. Similarly the section σ of the elliptic fibration $p : M \to S$ determines a Lagrangian section of $p : X \to C$.

$$\begin{array}{ccc} X & \xrightarrow{p} & C \\ \pi \downarrow & & \\ B & & \end{array}.$$

Given any elliptic fiber F in M, there is an one complex parameter family of points in M which intersect F, namely those points in F. Homologically speaking, suppose S is any coherent sheaf on M of the form $S = \iota_* L$ with $\iota : F \to M$ the natural inclusion and $L \in Pic^0(F) \simeq F^*$, the dual elliptic curve, then $Ext^*_{O_M}(\mathcal{S}, O_m) \neq 0$ exactly when $m \in F$. Translating this to the mirror side, given any Lagrangian fiber $\pi^{-1}(b)$, it should intersect an one real parameter family of $p^{-1}(c)$'s where $c \in C$.

On the other hand, given any point $m \in M$, there is a unique elliptic curve F that passes through m. But the coherent sheaves $S = \iota_* L$ with $L \in F^*$ also intersect m homologically and parametrized by an one complex parameter family, namely by F^*. On the mirror side, it says that given any Lagrangian fiber $p^{-1}(c)$ to $p : X \to C$, it should intersect an one real parameter family of $\pi^{-1}(b)$'s where $b \in B$.

Furthermore if $m \in M$ and $S_1, S_2 \in W$ satisfying $Ext^*_{O_M}(\mathcal{S}_1, O_m) \neq 0 \neq Ext^*_{O_M}(\mathcal{S}_2, O_m)$, then $m \in F_1 = F_2$ with $F_i = Supp S_i$. On the mirror side, this means that if $\pi^{-1}(b) \cap p^{-1}(c_i) \neq 0$ for $i = 1, 2$ then $\pi\left(p^{-1}(c_1)\right) = \pi\left(p^{-1}(c_2)\right)$. Similarly if $\pi^{-1}(b_i) \cap p^{-1}(c) \neq 0$ for $i = 1, 2$ then $p\left(\pi^{-1}(b_1)\right) = p\left(\pi^{-1}(b_2)\right)$. Namely $\pi\left(p^{-1}(c)\right)$'s in B form a fibration over some space D, which is also the base space of a fibration on C given by $p\left(\pi^{-1}(b)\right)$'s. That is,

$$\begin{array}{ccc} X & \to & C \\ \downarrow & & \downarrow \\ B & \to & D \end{array} .$$

We might also expect that these fibrations on B and C over D are both affine fibrations. Such a structure on X will be called a twin Lagrangian fibration on X. On the first sight, it seems that these two Lagrangian fibrations on X are on equal footing. But these arguments only valid outside singular fibers of $M \to S$. Recall that these two Lagrangian fibrations on X are mirror to holomorphic fibrations $id : M \to M$ and $p : M \to S$. The two base manifolds are quite different in nature: M is CY but S is not.

The Lagrangian fibers to $X \to C$ are mirror to the smooth elliptic curve fibers of $M \to S$. The situation near a singular elliptic curve fiber could be quite different. Their locus in S, called the discriminant locus $\mathcal{D}$, causes S fails to be CY because of the formula $K_S^{-1} = \frac{1}{12}\mathcal{D}$. In particular K_S^{-1} is an effective divisor on S, which is indeed ample in many cases, namely S is a Fano manifold.

There is a version of the mirror symmetry conjecture for Fano manifolds, and their mirror involve Lagrangian fibrations together with a holomorphic function, called the *superpotential*. It is reasonable to expect that the Lagrangian fibration $X \to C$ should also interact with this superpotential corresponding to S. We hope to come back to further discuss this issue in the future.

Large complex structure limits

Mirror symmetry for M is expected to work well near the large complex structure limit (LCSL). In terms of Hodge theory, it means that M is a member of an one parameter family of CY n-folds M_t with $0 < |t| < 1$ such that the monodromy operator $T : H^n(M) \to H^n(M)$ is of maximally unipotent, i.e. $N = \log(I - T)$ satisfies $N^n \neq 0$ but $N^{n+1} = 0$. On the mirror side [4], this corresponds to the Hard Lefschetz action $L = \wedge\omega_X : \oplus H^{p,p}(X) \to \oplus H^{p,p}(X)$ which satisfies $L^n \neq 0$ but $L^{n+1} = 0$.

We now assume that M has an elliptic fibration structure. In our above discussions, we need to require each member M_t in the family also have an elliptic fibration. When $n \geq 3$ the existence of an elliptic fibration is invariant under deformations of complex structures. In [26] Wilson showed that if M is a CY threefold then the existence of an elliptic fibration structure on M can be determined by cohomological condition. When $n = 2$ existence of an elliptic fibration is not a deformation invariant property because of $H^{2,0}(M) \neq 0$. The existence of an elliptic fibration on a K3 surface M is equivalent to finding $f \in H^2(M, \mathbb{Z}) \cap H^{1,1}(M)$ satisfying $f^2 = 0$. On the mirror side, this corresponds to the fact that the ω_X-Lagrangian fibration $X \to C$ on X is indeed Lagrangian with respect to $t\omega_X$ for every $t > 0$.

Calabi-Yau fibrations

Some of the discussions in this section work for the mirror X of a CY n-fold M with a fibration by q dimensional complex subvarieties. Every smooth fiber of such a fibration on M is automatically CY. In this case there will again be two Lagrangian fibrations on X with the property that each nonempty intersection of fibers of $X \to B$ and $X \to C$ should have codimension q in one of the fiber, and hence both.

We can also consider a CY manifold M with more than one fibration structures. For example, when M is a CY threefold with an elliptic fibration $M \to S$ over a Hirzebruch surface S. Then the $\mathbb{P}^1$-bundle structure on S gives a K3 fibration $M \to \mathbb{P}^1$ on M. In this circumstance, the mirror manifold X should admit three Lagrangian fibrations which are compatible to each other in certain large structure limit.

3 Fourier-Mukai Transform and Elliptic CY manifolds

3.1 General Fourier-Mukai Transform

Suppose M and W are smooth projective varieties. Given any coherent sheaf $\mathcal{P}$ on $M \times W$ we can define a Fourier-Mukai transform $\mathcal{F}$, or simply *FM transform*, between derived categories of coherent sheaves on M and W as follow [21],

$$\begin{aligned} \mathcal{F}_{cx}^{FM} &: D^b(M) \to D^b(W) \\ \mathcal{F}_{cx}^{FM}(-) &= R^\bullet p'_*(\mathcal{P} \otimes p^*(-)), \end{aligned}$$

where $p : M \times W \to M$ and $p' : M \times W \to W$ are projection maps. It was originally introduced by Mukai in the situation when M and W are dual Abelian varieties and $\mathcal{P}$ is the Poincaré bundle. In this case $\mathcal{F}_{cx}^{FM}$ is an equivalence of derived categories. We will need to use $\mathcal{F}_{cx}^{FM}$ for families of Abelian varieties situations. In general the FM transform is a useful tool to verify equivalences of derived categories, for example under flops or the McKay correspondence. On the other hand, a theorem of Orlov [21] says that any triangle-preserving equivalence $\Phi : D^b(M) \to D^b(W)$ is given by a FM transform.

3.2 Elliptic fibrations and their duals

In this subsection we recall basic facts about elliptic fibrations (see [6][7][9] for details). Suppose M is a CY manifold with an elliptic fibration $p : M \to S$ with section σ_M and with connected fibers. We denote its dual elliptic fibration as $p' : W \to S$, which is again an elliptic CY with a section σ_W. For example a CY hypersurface in a Fano toric variety which is a $\mathbb{P}^2$-bundle always admits an elliptic fibration structure.

Recall that the Weierstrass model (e.g. [1]) of a (reduced irreducible) elliptic curve T_τ in $\mathbb{P}^2$ is of the form $y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3$ with g_2, g_3 constants. The point $[0,1,0] \in \mathbb{P}^2$ is the origin of the elliptic curve T_τ. It is smooth if and only if $g_2^3 - 27g_3^2$ is nonzero. In our situation, we have a family of elliptic curves $p : M \to S$. We assume that all fibers are isomorphic to reduced irreducible cubic plane curves and S is smooth, then we can express M in a similar fashion with g_2 and g_3 vary over S. To describe it, we let $L = R^1 p_* O_M$ be a line bundle over S, which can also be identified as $O_M(-\sigma)|_\sigma$ where $\sigma = \sigma_M(S)$ is the given section. Then $g_2 \in H^0(S, L^{\otimes 4})$ and $g_3 \in H^0(S, L^{\otimes 6})$, and the Weierstrass model defines $M \subset \mathbb{P}(O_S \oplus L^{\otimes 2} \oplus L^{\otimes 3})$. Furthermore the discriminant locus in S is the zero locus of the section $g_2^3 - 27g_3^2$ of the bundle $L^{\otimes 12}$.

The dual elliptic fibration W is the compactified relative Jacobian of M, namely W parametrizes rank one torsion free sheaves of degree zero on fibers of $p : M \to S$. There is a natural identification between M and W because every elliptic curve T_τ is canonically isomorphic to its Jacobian $Jac(T_\tau)$ given by $T_\tau \to Jac(T_\tau)$, $p \longmapsto O(p - p_0)$ where p_0 is the origin of T_τ.

3.3 FM transform and spectral cover construction

Given any elliptic curve T_τ, or more generally a principally polarized Abelian variety, the dual elliptic curve T_τ^* is its Jacobian, which parametrizes topologically trivial holomorphic line bundles on T_τ. Mukai [20] used an analog of the Fourier transformation to define an equivalence of derived categories of coherent sheaves on T_τ and T_τ^*, $\mathcal{F}_{cx}^{FM} : D^b(T_\tau) \to D^b(T_\tau^*)$, called the *Fourier-Mukai transform.* This can be generalized to the family version as follow: Consider the relative Poincaré line bundle P on $M \times_S W$ which is given by

$$\mathcal{P} = O(\Delta - \sigma_M \times W - M \times \sigma_W),$$

where Δ is the relative diagonal in $M \times_S W$ and σ_M (resp. σ_W) is the section of $p : M \to S$ (resp. $p' : W \to S$). We define the following Fourier-Mukai functor $\mathcal{F}_{cx}^{FM} : D^b(M) \to D^b(W)$ as $\mathcal{F}_{cx}^{FM}(-) = R^{\bullet} p'_* (\mathcal{P} \otimes p^* (-))$, where the one in section 3.1 is a generalization of this. Then this can be proven to give an equivalence of derived categories [2][8][13]. Indeed

$$\begin{aligned}
\mathcal{F}_{cx,\mathcal{P}^* \otimes K_S}^{FM}\left(\mathcal{F}_{cx,\mathcal{P}}^{FM}(S)\right) &= S[-1] \\
\mathcal{F}_{cx,\mathcal{P}}^{FM}\left(\mathcal{F}_{cx,\mathcal{P}^* \otimes K_S}^{FM}(S)\right) &= S[-1].
\end{aligned}$$

Besides working on the level of derived categories, we can also study the FM transform of a stable bundle, the so-called *spectral cover construction* [6][7][9][10]. The basic idea is any stable bundle over an elliptic curve is essentially a direct sum of line bundles. In the family situation, a stable bundle on M gives a multi-section for $p' : W \to S$, together with a line bundle over it. Such construction is important in describing the moduli space of stable bundles and it can also be generalized to construct principal G-bundles on M, which play an important role in the duality between F-theory and String theory [9].

4 Symplectic FM Transform and Twin Lagrangian Fibrations

4.1 Symplectic Fourier-Mukai transform

Definition 1 *A* ***Lagrangian cycle*** *in a symplectic manifold (X, ω_X) is a pair $(L, \mathcal{L})$ with L a Lagrangian submanifold in X and $\mathcal{L}$ a unitary flat line bundle over L. We denote $\mathcal{C}(X)$ the collection of Lagrangian cycles in X.*

Lagrangian cycles are the objects that form the sophisticated Fukaya-Floer category $Fuk(X)$, where morphisms are Floer homology groups which counts holomorphic disks bounding cycles of Lagrangian submanifolds. We could also generalize the notion of Lagrangian cycles to allow L to be a stratified Lagrangian submanifold and to allow $\mathcal{L}$ to be a higher rank flat bundle.

Suppose (X, ω_X) and (Y, ω_Y) are symplectic manifolds of dimensions $2m$ and $2n$ respectively. Then $(X \times Y, \omega_X - \omega_Y)$ is again a symplectic manifold. Given any Lagrangian cycle $(P, \mathcal{P}) \in \mathcal{C}(X \times Y)$ we can construct a Fourier-Mukai type transformation, or simply *symplectic FM transform*, defined as follow [25],

$$\begin{gathered}
\mathcal{F}_{sym}^{FM} : \mathcal{C}(X) \to \mathcal{C}(Y) \\
\mathcal{F}_{sym}^{FM}(L, \mathcal{L}) = \left(\hat{L}, \hat{\mathcal{L}}\right).
\end{gathered}$$

Claim: The projection of $(L \times Y) \cap P \subset X \times Y$ to Y is a Lagrangian (immersed) submanifold in Y, which we denote it as $\hat{L}$.

First we suppose that $L \times Y$ intersects P transversely. For any point in L, by Darboux theorem, there exists a local coordinate $\left\{x^i, y^i\right\}_{i=1}^m$ centered at the point such that $\omega_X = \sum_{i=1}^m dx^i \wedge dy^i$ and L is given by $y^i = 0$ for all i. Suppose this point is also the X component of a transversal intersection point of $L \times Y$ and P. If we use an appropriate Darboux coordinates $\{u^k, v^k\}_{k=1}^n$ with $\omega_Y = \sum_{k=1}^n du^k \wedge dv^k$ and P is locally determined by

$$P : x^i = x^i(u, y) \text{ and } v^k = v^k(u, y) \text{ for all } i, k.$$

Then $(L \times Y) \cap P \subset X \times Y$ is locally given by

$$(L \times Y) \cap P : \begin{cases} x^i = x^i(u, 0), & y^i = 0, \\ u^k = u^k, & v^k = v^k(u, 0). \\ \text{(i.e. no restriction)} & \end{cases}$$

Therefore its projection to Y becomes

$$\hat{L} : \quad u^j = u^j \qquad v^k = v^k(u, 0).$$

The Lagrangian condition for $P \subset X \times Y$ is

$$\frac{\partial x^i}{\partial y^j} + \frac{\partial x^j}{\partial y^i} = 0 \text{ and } \frac{\partial v^k}{\partial u^l} + \frac{\partial v^l}{\partial u^k} = 0,$$

for any (u, y) and for all $i, j = 1, ..., m$ and $k, l = 1, ..., n$. By setting $y = 0$ for the second equation, this implies that $\hat{L}$ is a Lagrangian submanifold in Y.

Suppose $L \times Y$ and P do not intersect transversely, these arguments show that ω_Y vanishes on (smooth part of) $\hat{L}$. Even though the dimension of $(L \times Y) \cap P$ is larger than n, we have $\dim \hat{L} = n$ and $\hat{L}$ is a Lagrangian in Y.

To obtain the flat bundle $\hat{\mathcal{L}}$ over $\hat{L}$, in the generic case, we can identify $\hat{L} \subset Y$ with $(L \times Y) \cap P \subset X \times Y$. The flat bundle $\mathcal{L}$ on L induced one on $\hat{L}$ by pullback to $L \times Y$ and then restrict to $\hat{L}$. By restriction, $\mathcal{P}$ also determine a flat connection on $\hat{L}$. The tensor product of these two flat bundles is defined as $\hat{\mathcal{L}}$ over $\hat{L}$. More work will be needed to handle the pushforward in the non-generic case though.

Examples of Lagrangian cycles in products of symplectic manifolds include (i) the graph of any symplectic map $f : X \to Y$, i.e. $f^*\omega_y = \omega_X$; (ii) If M (resp. N) is a Lagrangian submanifold in X (resp. Y), then $P = M \times N$ is obviously a Lagrangian submanifold in $X \times Y$ and (iii) let C be any coisotropic submanifold in (X, ω_X). It induces a canonical isotropic foliation on C and such that its leaf space $C/\sim$ has a natural symplectic structure, provided that it is smooth and Hausdorff. This is called the *symplectic reduction.* Then the natural inclusion $C \subset X \times (C/\sim)$ is a Lagrangian submanifold in the product symplectic manifold. Thus we can use this to obtain a symplectic FM transform $\mathcal{F}_{sym}^{FM} : \mathcal{C}(X) \to \mathcal{C}(C/\sim)$.

4.2 Twin Lagrangian Fibrations

4.2.1 Review of Lagrangian fibrations

Suppose (X, ω) is a symplectic manifold with a Lagrangian fibration

$$\pi : X \to B$$

and with a Lagrangian section. Away from the singularities of π, we have an short exact sequence

$$0 \to T_{vert}X \to TX \to \pi^* TB \to 0,$$

where $T_{vert}X$ is the vertical tangent bundle. Using this and the Lagrangian condition, we have a canonical identification between $T_{vert}X$ and the pullback cotangent bundle,

$$T_{vert}X \cong \pi^* T^* B.$$

Therefore every cotangent vector at $b \in B$ determines a vector field on the fiber $X_b = \pi^{-1}(b)$. By integrating these constant vector fields on X_b, we obtain a natural *affine structure* on X_b. This implies that X_b must be an affine torus, i.e. $X_b = T_p X_b / \Lambda_b \cong T^b$ for some lattice Λ_b in $T_p X_b = T_b^* B$, provided that π is proper. This lattice structure on $T^* B$ in turn defines an affine structure on the base manifold B. Of course, this affine structure on B is only defined outside the discriminant locus, i.e. those $b \in B$ with X_b singular.

Before we discuss twin Lagrangian fibrations in details, let us first explain the linear aspects of them.

4.2.2 Linear algebra for twin Lagrangian fibrations

Suppose $\left(V \simeq \mathbb{R}^{2n}, \omega\right)$ is a symplectic vector space and T_b and T_c are two Lagrangian subspaces in V. They give two Lagrangian fibrations $V \to B$ and $V \to C$ with $B = V/T_b$ and $C = V/T_c$. If B and C intersect transversely, then there is a natural isomorphism

$$C \cong B^*$$

given by the following composition

$$C \hookrightarrow V \overset{\lrcorner\omega}{\to} V^* \twoheadrightarrow B^*.$$

Suppose $T_{bc} := T_b \cap T_c$ has dimension $n - q$ and we write $D = V/\left(T_b + T_c\right)$ then we have the following diagram of affine morphisms

$$\begin{array}{ccc} V & \to & C \\ \downarrow & & \downarrow \\ B & \to & D. \end{array}$$

The fibers of the columns (resp. rows) are T_b and T_b/T_{bc} (resp. T_c and T_c/T_{bc}). The Lagrangian conditions implies the existence of a natural isomorphism $T_c/T_{bc} \cong \left(T_b/T_{bc}\right)^*$, namely the two affine bundles $B \to D$ and $C \to D$ are fiberwise dual to each other.

The usual SYZ transform which switches the fibers of a Lagrangian fibration $V \to B$ to their duals will interchanges complex geometry and symplectic geometry. In order to stay within the symplectic geometry, we should take the fiberwise dual to both Lagrangian fibrations $V \to B$ and $V \to C$.

Taking dual to both T_b and T_c has the same effect as taking dual to $(T_b + T_c)/T_{bc}$ while keeping T_{bc} fixed. This gives us the following new commutative diagram.

$$\begin{array}{ccc} U & \to & C' \\ \downarrow & & \downarrow \\ B' & \to & D. \end{array}$$

Here B' (resp. C') is the total space of taking fiberwise dual to the fibration $B \to D$ (resp. $C \to D$). The fiber T_b' of $U \to B'$ is obtained by taking dual along the base of the fibration $T_b \to T_b/T_{bc}$. This is the same as taking fiberwise dual, up to conjugation with the duality of total spaces. The fiber of $B' \to D$ is $(T_c/T_{bc})^*$ and likewise for $C' \to D$.

A more intrinsic way to describe this double dual process is as follow: The fiber of $V \to D$ is a coisotropic subspace in V, say V_d. The symplectic reduction $V_d/\sim$ (see e.g. [25]) is another symplectic vector space. Then U is obtained by replacing $V_d/\sim$ by its dual symplectic space from V.

In terms of an explicit coordinate system on V given by $\left\{x^i, x^\alpha, y_i, y_\alpha\right\}$ with $1 \leq i \leq n-q$ and $n-q+1 \leq \alpha \leq n$, then we have,

$$\begin{array}{ccc} V & \to & C \\ \downarrow & & \downarrow \\ B & \to & D \end{array} \text{ with coordinates } \begin{array}{ccc} \begin{array}{c} y_i, y_\alpha \\ x^\alpha, x^i \end{array} & \to & \begin{array}{c} y_\alpha \\ x^i \end{array} \\ \downarrow & & \downarrow \\ x^\alpha, x^i & \to & x^i \end{array},$$

and

$$\begin{array}{ccc} U & \to & C' \\ \downarrow & & \downarrow \\ B' & \to & D \end{array} \text{ with coordinates } \begin{array}{ccc} \begin{array}{c} y_i, y_\alpha^* \\ x^{\alpha *}, x^i \end{array} & \to & \begin{array}{c} y_\alpha^* \\ x^i \end{array} \\ \downarrow & & \downarrow \\ x^{\alpha *}, x^i & \to & x^i \end{array}.$$

The Lagrangian conditions actually imply that $B' \cong C$, $C' \cong B$ and $U \cong V$. We now return back to the general symplectic manifolds situation.

4.2.3 Twin Lagrangian fibrations

First we recall that every fiber of a Lagrangian fibration, say $\pi : X \to B$, has a natural affine structure.

Definition 2 *Let (X, ω) be a symplectic manifold of dimension $2n$ and $\pi : X \to B$ and $p : X \to C$ are two Lagrangian fibrations on X with Lagrangian sections. We call this a twin Lagrangian fibration of index q if for general $b \in B$ and $c \in C$ with $p^{-1}(c) \cap \pi^{-1}(b)$ nonempty, then it is an affine subspace of $\pi^{-1}(b)$ of codimension q. We denote such a structure as $B \overset{\pi}{\leftarrow} X \overset{p}{\to} C$*

Since $\pi^{-1}(b)$ is the union of affine subspaces $p^{-1}(c) \cap \pi^{-1}(b)$, they form an affine foliation of $\pi^{-1}(b)$ of codimension q. In the above definition we assume that when $p^{-1}(c) \cap \pi^{-1}(b)$ is nonempty, then it is a codimension q affine submanifold of $\pi^{-1}(b)$. We did not assume that this affine structure is compatible with the one on $p^{-1}(c)$ which comes from the other Lagrangian fibration p. Nevertheless, under suitable assumptions, we will show that this is indeed the case and the above definition is symmetric with respect to B and C. Furthermore we have a commutative diagram of affine morphisms.

$$\begin{array}{ccc} X & \rightarrow & C \\ \downarrow & & \downarrow \\ B & \rightarrow & D \end{array},$$

Claim 3 *Suppose that $B \overset{\pi}{\leftarrow} X \overset{p}{\rightarrow} C$ is a twin Lagrangian fibration. Then B admits a natural (singular) foliation whose leaves are q dimensional subspaces $\pi\left(p^{-1}(c)\right)$'s with $c \in C$.*

Proof: Since $p^{-1}(c) \cap \pi^{-1}(b)$ is always of codimension k in $p^{-1}(c)$ if nonempty, $\pi\left(p^{-1}(c)\right)$ is a q dimensional subspace in B. Suppose b is a smooth point in $\pi\left(p^{-1}(c)\right)$, we claim that its tangent space $T_b\left(\pi\left(p^{-1}(c)\right)\right) \subset T_bB$ is independent of the choice of c. Assuming this, we obtain a q dimensional (singular) distribution on B. Furthermore $\pi\left(p^{-1}(c)\right)$'s with $c \in C$ are leaves of this distribution, thus we have the required foliation on B.

To prove the claim, we recall that $\pi^{-1}(b)$ has a natural affine structure and $\pi^{-1}(b)$ admits an affine foliation whose leaves are $p^{-1}(c) \cap \pi^{-1}(b)$ with $c \in C$, by the assumption of a twin Lagrangian fibration. The key observation is these imply that for any $x \in p^{-1}(c) \cap \pi^{-1}(b)$, under the natural identification $T_x^*\left(\pi^{-1}(b)\right) \simeq T_bB$, the conormal bundle of $p^{-1}(c) \cap \pi^{-1}(b)$ in $\pi^{-1}(b)$ at x is the same linear subspace in T_bB of dimension q. Furthermore this coincides with $\pi\left(T_x\left(p^{-1}(c)\right)\right)$. Hence the result. ■

We will assume that this foliation on B is indeed a fibration which we denote as

$$\overline{p} : B \rightarrow D.$$

As a corollary of the above claim, we have the following immediate result.

Corollary 4 *Suppose that $B \overset{\pi}{\leftarrow} X \overset{p}{\rightarrow} C$ is a Lagrangian twin fibration. Then we have a commutative diagram*

$$\begin{array}{ccc} X & \rightarrow & C \\ \downarrow & & \downarrow \\ B & \rightarrow & D \end{array},$$

where $\overline{p} : B \rightarrow D$ is the above fibration fibered by $\pi\left(p^{-1}(c)\right)$'s on B.

Claim 5 *Suppose that $B \overset{\pi}{\leftarrow} X \overset{p}{\rightarrow} C$ is a twin Lagrangian fibration with sections and π is a proper map. Then D has a natural affine structure and $\overline{p} : B \rightarrow D$ is an affine morphism.*

Proof: Since $\pi : X \to B$ is a Lagrangian fibration, its general fiber $\pi^{-1}(b)$ has a natural affine structure, thus as an affine manifold $\pi^{-1}(b) \simeq R^n/\Lambda$ for some lattice $\Lambda \simeq Z^r$ in T_b^*B. We have $r = n$ because π is proper, i.e. $\pi^{-1}(b)$ is compact. This full rank lattice $\Lambda \subset T_b^*B$ determines an integral affine structure on B, away from the locus of singular fibers. Thus the proposition is equivalent to the fibration of $\pi\left(p^{-1}(c)\right)$'s being an affine fibration on B.

From the proof of the previous proposition, we know that $p^{-1}(c) \cap \pi^{-1}(b)$ is an affine subtorus of $\pi^{-1}(b) \simeq T^n$. This implies that $T_b\left(p^{-1}(c)\right) \subset T_bB$ is an integral affine subspace. Because of the integral structure, these subspaces $T_b\left(p^{-1}(c)\right)$'s are affinely equivalently to each other locally. Hence the foliation they determine is an affine foliation in B. In particular the leave space D inherits an affine structure such that $\overline{p} : B \to D$ is an affine morphism. Hence the result. ■

This implies that, outside the singular locus of $\overline{p} : B \to D$, there exists a rank q vector bundle $\mathbb{R}^q \to E \xrightarrow{\varepsilon} D$ with affine gluing functions, together with a multi-section $E_{\mathbb{Z}}$ such that $\overline{p}$ is affine isomorphic to the projection $E/E_{\mathbb{Z}} \to D$. In particular $B = E/E_{\mathbb{Z}}$.

Given a general $x \in X$ with $b = \pi(x)$, $c = p(x)$ and $d = \overline{p}(b) = \overline{\pi}(b)$, we write $X_d = p^{-1}(c) \cap \pi^{-1}(b)$. From π and p both being Lagrangian fibrations on X, we have

$$0 \to N^*_{X_d/\pi^{-1}(b),x} \to T_bB \to T_x^*X_d \to 0,$$

and

$$0 \to N^*_{X_d/p^{-1}(c),x} \to T_cC \to T_x^*X_d \to 0.$$

As we have shown earlier $N_{X_d/p^{-1}(c),x}$ can be identified with the fiber of $E \to D$ over $d \in D$, denoted as E_d. Moreover the exact sequence $0 \to N^*_{X_d/\pi^{-1}(b),x} \to T_bB \to T_x^*X_d \to 0$ is equivalent to $0 \to \varepsilon^*E \to TE \to \varepsilon^*TD \to 0$ for the vector bundle $\varepsilon : E \to D$. This implies that T_cC is naturally isomorphic to the tangent space of E^* at d.

Notice that the affine structure of the fibers of the Lagrangian fibration $p : X \to C$ is given by $T\left(p^{-1}(c)\right) \simeq p^*\left(T_c^*C\right)$. When p is proper, these data also determine an affine structure on C. Thus having such a natural identification between T_cC and $T_d(E^*)$, all the affine structures involved are compatible with each other. Such a claim can be checked by a direct diagram chasing method. Thus we have obtained the following result.

Claim 6 *Suppose $B \xleftarrow{\pi} X \xrightarrow{p} C$ is a twin Lagrangian fibration with π and p proper. Then $C \xleftarrow{p} X \xrightarrow{\pi} B$ is also a twin Lagrangian fibration and*

$$\begin{array}{ccc} X & \to & C \\ \downarrow & & \downarrow \\ B & \to & D \end{array}$$

is a commutative diagram of affine morphisms.

Similar to $B = E/E_{\mathbb{Z}}$ for a vector bundle $E \to D$, we have $C = F/F_{\mathbb{Z}}$ for another vector bundle $F \to D$ of rank q and multisection $F_{\mathbb{Z}}$ of it. Furthermore the two bundles E and F over D are fiberwise dual to each other.

Similar discussions can be applied to Lagrangian sections to $\pi : X \to B$ and $p : X \to C$ and we can obtain affine sections to affine fibrations $B \to D$ and $C \to D$ and we can also prove that the following diagram of sections is commutative,

$$\begin{array}{ccc} X & \leftarrow & C \\ \uparrow & & \uparrow \\ B & \leftarrow & D \end{array}.$$

Remark: The composition map $X \to D$ is a coisotropic fibration with fiber dimension equals $n+q$. If we apply the symplectic reduction on each coisotropic fiber, then we obtain a symplectic fibration over D which can be naturally identified with $B \times_D C = (E \oplus F) / (E_{\mathbb{Z}} \oplus F_{\mathbb{Z}})$.

Conversely, suppose that we have two proper Lagrangian fibrations $\pi : X \to B$ and $p : X \to C$ and a commutative diagram of maps

$$\begin{array}{ccc} X & \to & C \\ \downarrow & & \downarrow \\ B & \to & D_0 \end{array},$$

for some space D_0 satisfying for any $b \in B$ and $c \in C$ with the same image $d \in D_0$, then the preimage of d in X for the composition map is equal to $\pi^{-1}(b) \cap p^{-1}(c)$. We leave it as an exercise for readers to show that this gives a twin Lagrangian fibration structure on X.

4.3 Dual twin Lagrangian fibrations and its FM transform

Suppose (X, ω) is a symplectic manifold with a twin Lagrangian fibration with sections.

$$\begin{array}{ccc} X & \to & C \\ \downarrow & & \downarrow \\ B & \to & D \end{array}$$

The dual of such a structure is obtained by taking the fiberwise dual to both fibrations $X \to B$ and $X \to C$ and we obtain the following commutative diagram.

$$\begin{array}{ccc} Y & \to & C' \\ \downarrow & & \downarrow \\ B' & \to & D \end{array}.$$

Recall that a general fiber of $X \to D$ is a coisotropic submanifold in X which is a torus of dimension $n+q$, denoted T_d. Its symplectic reduction $(T_d/\sim)$ is a symplectic torus of dimension $2q$. We can view Y as being obtained from X by replacing those directions along the symplectic torus $(T_d/\sim)$ by the dual symplectic torus $(T_d/\sim)^*$. It is an important problem to describe Y near singular fibers. Such a Y is called the *dual twin Lagrangian fibration* to X. It is clear that the dual twin Lagrangian fibration to Y is X again.

As we have explained in the linear situation, the fibration $B' \to D$ (resp. $C' \to D$) is the dual fibration to $B \to D$ (resp. $C \to D$). Furthermore the Lagrangian conditions imply that there are natural identifications $B' \cong C$ and $C' \cong B$ and also

$$X \cong Y.$$

It is interesting to know whether this identification will continue to hold true if superpotentials are also included in our discussions.

Since $X \cong Y$, we choose the Lagrangian cycle $\left(P_{sym}^{FM}, \mathcal{P}_{sym}^{FM}\right)$ on $X \times Y$ given by the diagonal Lagrangian submanifold together with the trivial flat bundle over it. We call this the *Lagrangian Poincaré cycle* on $X \times Y$. The symplectic FM transform $\mathcal{F}_{sym}^{FM}$ for the twin Lagrangian fibration on X and its dual twin Lagrangian fibration on Y is defined using this Lagrangian cycle.

$$\mathcal{F}_{sym}^{FM} : \mathcal{C}(X) \to \mathcal{C}(Y)$$
$$\mathcal{F}_{sym}^{FM}(L, \mathcal{L}) = \left(\hat{L}, \hat{\mathcal{L}}\right).$$

Notice that this is actually an identity transformation.

4.4 Examples of Twin Lagrangian fibrations

We are going describe some examples of symplectic manifolds X with twin Lagrangian fibrations. First we notice that this property is stable under taking the product with another symplectic manifold with a Lagrangian fibration.

The trivial example is the product of a flat torus with its dual, $T \times T^*$. The complex projective space $\mathbb{CP}^n$ also has a twin Lagrangian fibration: its toric fibration

$$\mu : \mathbb{CP}^n \to \mathbb{R}^n$$
$$\mu[z_0, ..., z_n] = \left(\frac{|z_1|^2}{\sum_{j=0}^n |z_j|^2}, \frac{|z_2|^2}{\sum_{j=0}^n |z_j|^2}, ..., \frac{|z_n|^2}{\sum_{j=0}^n |z_j|^2}\right).$$

is a Lagrangian fibration. If we consider an automorphism of $\mathbb{CP}^n$ defined by $f([z_0, ..., z_n]) = [z_0 + z_1, z_0 - z_1, z_2, ..., z_n]$, then

$$\mu \circ f : \mathbb{CP}^n \to \mathbb{R}^n,$$

gives another Lagrangian fibration on $\mathbb{CP}^n$. It is easy to check that the nontrivial intersection of any two generic fibers of μ and $\mu \circ f$ is an $(n-1)$-dimensional torus T^{n-1}. This example can be generalized to the Gelfand-Zeltin system on Grassmannians and partial flag varieties.

Taub-NUT example: Besides trivial examples of products of Lagrangian fibrations, we can write down explicit twin Lagrangian fibrations on four dimensional hyperkähler manifolds X with S^1-symmetry. These spaces are classified by an positive integer n, denoted A_n, with explicit Taub-NUT metrics. Topologically A_n is the canonical resolution of $\mathbb{C}^2/\mathbb{Z}_{n+1}$ for a finite subgroup $\mathbb{Z}_{n+1} \subset SU(2)$. The hyperkähler moment map

$$\mu = (\mu_1, \mu_2, \mu_3) : X \to \operatorname{Im} \mathbb{H} = \mathbb{R}^3,$$

is a singular S^1-fibration on X. For any unit vector $t \in S^2 \subset \mathbb{R}^3$, it determines a complex structure J_t on X. Furthermore the composition of μ with the projection to the orthogonal plane to t gives a Ω_{J_t}-complex Lagrangian fibration on X.

Corresponding to $i, j, k \in \operatorname{Im} \mathbb{H}$ we have complex structures I, J, K on X. Thus $(\mu_2, \mu_3) : X \to \mathbb{R}^2 = \mathbb{C}$ is a Ω_I-complex Lagrangian fibration with coordinate $(y, z) \in \mathbb{C}$. Similarly $(\mu_1, \mu_3) : X \to \mathbb{R}^2 = B$ is a Ω_J-complex Lagrangian fibration with coordinate $(x, z) \in B$. Then we have a ω_K-twin Lagrangian fibration structure on X with D being the z-axis.

This construction can be easily generalized to any $4n$ dimensional hyperkähler manifold X with a tri-Hamiltonian T^n action with a hyperkähler moment map

$$\mu : X \to \mathbb{R}^n \otimes \mathbb{R}^3.$$

In the above example, the generic fiber of $X \to B$, or $X \to C$, is topologically a cylinder, thus noncompact. Therefore the base of the Lagrangian fibration only have a partial affine structure. For Lagrangian fibrations with compact fibers, there are usually singularities for the affine structures for the base spaces corresponding to singular fibers. When $\dim B = 2$, the affine structure near a generic singularity has been studied in [12][15]. The monodromy across the 'slit', namely the positive x-axis, is given by $(x, y) \to (x, x + y)$. Near the singular point $0 \in B = \mathbb{R}^2$, the only affine fibration is given by horizontal lines. Other attempts to obtain fibrations on B will get overlapping fibers.

K3 surface with an elliptic fibration $M \to \mathbb{P}^1$ can be constructed as an anticanonical divisor of a Fano toric threefold P_Δ which admits a toric $\mathbb{P}^2$-bundle structure, $\mathbb{P}^2 \to P_\Delta \to \mathbb{P}^1$. The mirror to M is another K3 surface X inside a Fano toric threefold P_∇ associated to a polytope which is a 2-sided cone over a triangle.

Generically $X = \{f = 0\}$ admits a Lagrangian fibration over $B = \partial\nabla$, homeomorphic to the two sphere S^2, with 24 singular fibers. B admits a natural affine structure with singularity along the base points of these singular fibers (see [12] for its construction), which all lie on the edges of ∇, and their 'slits' are also along these edges. We consider the restriction of the projection of $\mathbb{R}^3$ to the x-axis to $B = \partial\nabla$, the image is the interval D in $\mathbb{R}$ corresponding to the polytope of the base $\mathbb{P}^1$,

$$p : B \to D.$$

Since the slits for all the singular points in the middle triangle in $\partial\nabla$ are vertical with respect to p, we obtain an affine fibration around there. But p will cease to be an affine fibration around other singular points of B. However if we choose the defining function f for M appropriately, then we can arrange all other singular points to be far away from this middle triangle. Thus we obtain an affine fibration on a large portion of B. As a matter of fact, if we allow M to be singular, then we can make p to be an affine fibration on the whole $B \backslash Sing\,(B)$. This picture can be generalized to certain higher dimensional manifolds, for instance CY hypersurfaces in toric varieties.

5 SYZ Transformation of FM transform

Given an elliptically fibered CY manifold M, we argued in section 2.4 that its mirror manifold X should admit a twin Lagrangian fibration (with superpotential which we neglect in our present discussions). In this section, we first argue that the mirror manifold to the dual elliptic fibration W to M is the dual twin Lagrangian fibration Y to X. Second we will explain the mirror to the universal Poincaré sheaf that defines the FM transform between complex geometries of M and W is the universal Lagrangian Poincaré cycle that defines the FM transform between symplectic geometries of X and Y. Third we show that these FM transforms commute with the SYZ transforms. Namely the SYZ transform of the complex FM transform is the symplectic FM transform, which is actually an identity transformation.

5.1 SYZ transform of dual elliptic fibrations

As we mentioned above, we are going to argue that the following diagram commutes.

	M	$\xleftrightarrow{\text{mirror manifolds}}$	X	
dual elliptic fibrations	$\updownarrow$		$\updownarrow$	dual twin Lagr fibrations
	W	$\xleftrightarrow{\text{mirror manifolds}}$	Y	

Indeed our arguments only work in the large structure limit but we expect that a modified version of such will work in more general situations. To be precise, we are only dealing with the flat situation: We assume that M is Calabi-Yau n-fold with a q-dimensional Abelian varieties fibration $p : M \to S$. We are going to impose a very restrictive assumption, which we expect to be the first order approximation of what happens at the large structure limit. Namely M is the product of an Abelian variety with S.

Let $z^1, ..., z^n$ be a local complex coordinate system on M such that $z = x + \sqrt{-1}y$ with $y^1, ..., y^n$ (resp. $x^1, ..., x^n$) be an affine coordinate system on fibers (resp. base) of the special Lagrangian fibration on $M \to B$. According to the SYZ proposal, the mirror manifold X to M is the total space of the dual Lagrangian torus fibration. For the affine coordinate system $y^1, ..., y^n$ on the torus fiber, we denote the dual coordinate system on its dual torus as $y_1, ..., y_n$. Thus X has a Darboux coordinate system given by $y_1, ..., y_n$, $x^1, ..., x^n$.

We assume the q-dimensional Abelian variety fibration $M \to S$ is compatible with the Lagrangian fibration structure in the sense that $z^{n-q+1}, ..., z^n$ gives a complex affine coordinate system on fibers to $M \to S$. In particular, the dual coordinate system on the dual Abelian variety is given by $z_{n-q+1}, ..., z_n$. Therefore such a coordinate system on M induces a similar coordinate system on the dual Abelian variety fibration $W \to S$, labelled $\left(z^1, ..., z^{n-q}, z_{n-q+1}, ..., z_n\right)$.

Note that on a fiber of the Abelian variety fibration $M \to S$, $y^{n-q+1}, ..., y^n$ (resp. $x^{n-q+1}, ..., x^n$) are those coordinates belong to Lagrangian fibers (resp. base) for the special Lagrangian fibration on M. Since fibers are much smaller in scale when comparing to the base near the large complex structure limit [22], when we take the dual Abelian variety, $y_{n-q+1}, ..., y_n$ will become much larger in scale when comparing to $x_{n-q+1}, ..., x_n$. As a consequences, the special Lagrangian fiber (resp. base) coordinates for the CY manifold Y are $y^1, ..., y^{n-q}$,$x_{n-q+1}, ..., x_n$ (resp. $x^1, ..., x^{n-q}, y_{n-q+1}, ..., y_n$). Therefore the special Lagrangian fibration on the mirror manifold Y to W has fiber (resp. base) coordinates as $y_1, ..., y_{n-q}, x^{n-q+1}, ..., x^n$ (resp. $x^1, ..., x^{n-q}, y_{n-q+1}, ..., y_n$).

Next we need to check that Y is indeed the total space of the dual twin Lagrangian fibration to X. For the Abelian variety fiber in M, with coordinates $x^{n-q+1}, ...x^n, y^{n-q+1}, ...y^n$, its mirror Lagrangian cycle in X has coordinates $x^{n-q+1}, ...x^n, y_1, ...y_{n-q}$ and it is the fiber of the other Lagrangian fibration on X. Similarly, for the Abelian variety fibers in W, with coordinates $x_{n-q+1}, ...x_n, y_{n-q+1}, ...y_n$, its mirror Lagrangian cycle in Y has coordinates $y_1, ...y_{n-q}, y_{n-q+1}, ...y_n$ and it is the fiber of the other Lagrangian fibration on Y. It is now clear that X and Y are dual twin Lagrangian fibrations. All these fiber/base coordinates systems are summarized in the following diagram:

$$\begin{array}{ccccc} M: \begin{matrix} y^i & y^\alpha \\ x^i & x^\alpha \end{matrix} & & \longleftrightarrow & & X: \begin{matrix} y_\alpha & y_i \\ x^i & x^\alpha \end{matrix} \\ \updownarrow & & & & \updownarrow \\ W: \begin{matrix} y^i & x_\alpha \\ x^i & y_\alpha \end{matrix} & & \longleftrightarrow & & Y: \begin{matrix} x^\alpha & y_i \\ x^i & y_\alpha \end{matrix} \end{array}$$

Our conventions are $i = 1, ..., n-q$ and $\alpha = n-q+1, ..., n$. For each space in the above diagram, the coordinates in the top (resp. bottom) row are for the SYZ special Lagrangian fibers (resp. base). Also the coordinates in the left (resp. right) column are for the fibers (resp. base) of the Abelian variety fibrations for M or W and the other Lagrangian fibrations for X or Y.

5.2 SYZ transform of the Universal Poincaré sheaf

Recall that the FM transform on derived categories of manifolds with elliptic fibrations is given by

$$FM : D^b(M) \to D^b(W)$$
$$FM(-) = R^\bullet p'_*\left(\mathcal{P}_{cx} \otimes p^*(-)\right),$$

where $\mathcal{P}_{cx}$ is a coherent sheaf on $M \times W$ with support

$$Supp\left(\mathcal{P}_{cx}\right) = M \times_S W \subset M \times W,$$

and

$$\mathcal{P}_{cx} = O\left(\Delta - \sigma_M \times W - M \times \sigma_W\right),$$

with Δ the relative diagonal in $M \times_S W$.

We note that $M \times W$ is again a CY manifold with its mirror manifold being $X \times Y$. We are going to describe the SYZ transform of $\mathcal{P}_{cx}$ from $D^b(M \times W)$ to $Fuk(X \times Y)$. We will apply the transformation on the level of object, as proposed in [22].

Claim 7 *Corresponding to the mirror symmetry between Calabi-Yau $(2n)$-folds $M \times W$ and $X \times Y$,*

$$\begin{matrix}\textit{Complex}\\ \textit{Geometry}\end{matrix}\left((M \times W)\right) \quad \xleftrightarrow{\textit{SYZ transform}} \quad \begin{matrix}\textit{Symplectic}\\ \textit{Geometry}\end{matrix}\left((X \times Y)\right),$$

the mirror of the coherent sheaf $\mathcal{P}_{cx}$ on $M \times W$ is the diagonal Lagrangian cycle $(P_{sym}, \mathcal{P}_{sym}) \in \mathcal{C}(X \times Y)$ as described in section 4.3.

Proof: We will continue to use the same coordinate systems as before. The Poincaré sheaf $\mathcal{P}_{cx}$ for the dual Abelian variety fibrations $M \to S$ and $W \to S$ has support $M \times_S W \subset M \times W$ which has coordinates $\left\{x^i, y^i, x^\alpha, y^\alpha, x_\alpha, y_\alpha\right\}_{i,\alpha}$. (where x^i and y^i are diagonal coordinates in the product space). Over $M \times_S W$, $\mathcal{P}_{cx}$ is indeed a complex line bundle with an $U(1)$ connection:

$$D_{cx}^{FM} = d + \sqrt{-1} \sum_\alpha \left(x_\alpha dx^\alpha + x^\alpha dx_\alpha - y_\alpha dy^\alpha - y^\alpha dy_\alpha\right).$$

The reason that the signs for terms involving x and y are different is the following: For Abelian varieties, the dual *complex* coordinates are z^α and z_α, which induces the dual coordinates for x^α, y^α as $x_\alpha, -y_\alpha$ because $\operatorname{Re}\left(z^\alpha z_\alpha\right) = x^\alpha x_\alpha - y^\alpha y_\alpha$.

To apply the SYZ mirror transform to $\mathcal{P}_{cx}$ from the special Lagrangian fibration $M \times W \to B \times B^*$ to the one $X \times Y \to B \times B^*$, we first need to describe the Poincaré bundle $\mathcal{P}^{SYZ}$ over

$$(M \times W) \underset{B \times B^*}{\times} (X \times Y) \subset (M \times W) \times (X \times Y).$$

To describe the coordinates on this space, we need to rename the (x, y) coordinates on W and Y to (u, v) coordinates. That is,

$$\begin{array}{ccccc} & W & & Y & \\ v^i & u_\alpha & \overset{\text{SYZ mirror}}{\longleftrightarrow} & u^\alpha & v_i \\ u^i & v_\alpha & & u^i & v_\alpha \end{array}$$

Now the universal $U(1)$ connection on the line bundle

$$\mathbb{C} \rightarrow \mathcal{P}^{SYZ} \rightarrow (M \times W) \underset{B \times B^*}{\times} (X \times Y)$$

is given by

$$\begin{aligned} \mathcal{D}^{SYZ} &= d + \sqrt{-1} \sum \left(y^i dy_i + y_i dy^i + y^\alpha dy_\alpha + y_\alpha dy^\alpha \right) \\ &\quad - \sqrt{-1} \sum \left(v^i dv_i + v_i dv^i + u_\alpha du^\alpha + u^\alpha du_\alpha \right). \end{aligned}$$

Note that we have used different signs for the SYZ transform between M and X and SYZ transform between W and Y.

In this newly named coordinates on W, The universal connection on the Poincaré bundle

$$\mathbb{C} \rightarrow \mathcal{P}_{cx}^{FM} \rightarrow M \times_S W,$$

is given by

$$\mathcal{D}_{cx}^{FM} = d - \sqrt{-1} \sum_\alpha \left(u_\alpha dx^\alpha + x^\alpha du_\alpha - v_\alpha dy^\alpha - y^\alpha dv_\alpha \right).$$

To apply the SYZ transform, we need to first pullback the bundle $\mathcal{P}_{cx}^{FM}$ (with its connection $\mathcal{D}_{cx}^{FM}$) from $M \times_S W \subset M \times W$ to $(M \times W) \times_{B \times B^*} (X \times Y) \subset (M \times W) \times (X \times Y)$ and tensors it with $\mathcal{P}^{SYZ}$ (with its connection $\mathcal{D}^{SYZ}$). Then we pushforward along the projective map $(M \times W) \times_{B \times B^*} (X \times Y) \rightarrow X \times Y$.

We can separate our discussions into two parts: (i) perform the SYZ transform along those directions with indexes $i = 1, ..., n - q$ and (ii) perform the SYZ transform along those directions with indexes $\alpha = n - q + 1, ..., n$.

Part (i) with $i = 1, ..., n - q$. This part is easy because the Poincaré bundle for the FM transform does not involve here. In fact it is the trivial line bundle over $\left\{u^i = x^i\right\} \cap \left\{v^i = y^i\right\} \subseteq M \times W$. In these coordinates, the SYZ Poincaré bundle $\mathcal{P}^{SYZ}$ has support $(M \times W) \times_{B \times B^*} (X \times Y) \subset (M \times W) \times (X \times Y)$, which in our coordinate systems means the x^i's coordinates for M and X are the same and similarly the u^i's coordinates for W and Y are the same.

$$\begin{aligned} \mathcal{D}^{SYZ} &= d + \sqrt{-1} \sum_i y^i dy_i + y_i dy^i - v^i dv_i - v_i dv^i \\ &= d + \sqrt{-1} \sum_i y^i \left(dy_i - dv_i \right) + \left(y_i - v_i \right) dy^i \end{aligned}$$

To pushforward to $X \times Y$, we integrate along y^i's directions. Along these directions, the restriction of the above connection is $d + \sqrt{-1} \sum_i \left(y_i - v_i \right) dy^i$, which has a (unique up to scaling) flat section precisely when $y_i - v_i = 0$ for all i. When this happens, we also have $y^i (dy_i - dv_i) = 0$. Hence the SYZ transform of $\mathcal{P}_{cx}^{FM}$ is given by the trivial bundle over $\{u^i = x^i \text{ and } y_i = v_i \text{ for all } i\} \subset X \times Y$. It is a Lagrangian submanifold in $X \times Y$ with the symplectic form $\omega_{X\times Y} = \Sigma dx^i \wedge dy_i + \Sigma du^i \wedge dv_i$.

Part (ii) with $\alpha = n - q + 1, ..., n$. In these coordinates, the support for the SYZ Poincaré bundle $\mathcal{P}^{SYZ}$ imposes a constraint which says that the x^α's coordinates for M and X are the same and the v_α's coordinates for W and Y are the same. Now we need to integrate the directions y^α's and u_α's. First we rearrange the terms

$$\begin{aligned} \mathcal{D}_{cx}^{FM} \otimes \mathcal{D}^{SYZ} &= d - \sqrt{-1} \textstyle\sum_\alpha (u_\alpha dx^\alpha + x^\alpha du_\alpha - v_\alpha dy^\alpha - y^\alpha dv_\alpha) \\ &\quad + \sqrt{-1} \textstyle\sum_\alpha (y^\alpha dy_\alpha + y_\alpha dy^\alpha - u_\alpha du^\alpha - u^\alpha du_\alpha) \\ &= d + \sqrt{-1} \textstyle\sum_\alpha ((-v_\alpha + y_\alpha) dy^\alpha + (x^\alpha - u^\alpha) du_\alpha) \\ &\quad + \sqrt{-1} \textstyle\sum_\alpha (u_\alpha dx^\alpha - y^\alpha dv_\alpha + y^\alpha dy_\alpha - u_\alpha du^\alpha). \end{aligned}$$

When we integrate along the y^α's and u_α's directions, the terms $u_\alpha dx^\alpha$, $y^\alpha dv_\alpha$, $y^\alpha dy_\alpha$ and $u_\alpha du^\alpha$ has nontrivial Fourier modes in these variables and therefore they contribute zero to $\mathcal{P}_{sym}$, the mirror cycle of $\mathcal{P}_{cx}$.

When we pushforward along y^α's direction, the restriction of the above connection becomes $d + \sqrt{-1} \sum_\alpha (-v_\alpha + y_\alpha) dy^\alpha$ along any fiber and it admits a (unique up to scaling) parallel section precisely when $y_\alpha = v_\alpha$ for all α. Similarly we obtain $u^\alpha = x^\alpha$ for all α when we pushforward along u_α's directions. Namely the support of the mirror object to $\mathcal{P}_{cx}^{FM}$ is given by

$$\{y_\alpha = v_\alpha \text{ and } x^\alpha = u^\alpha \text{ for all } \alpha\} = P_{sym} \subset X \times Y.$$

Namely this is precisely given by the diagonal Lagrangian submanifold as in the example in section 4.3.

There is no remaining component of $\mathcal{D}_{cx}^{FM} \otimes \mathcal{D}^{SYZ}$ and therefore the unitary flat connection on $\mathcal{P}_{sym}$ is trivial.

By putting parts (i) and (ii) together, we have shown that the mirror transformation of the complex Poincaré cycle $\left(P_{cx}^{FM} = M \times_S W, \mathcal{P}_{cx}^{FM}\right) \in \mathcal{C}(M \times W)$ is the Lagrangian Poincaré cycle $\left(P_{sym}^{FM}, \mathcal{P}_{sym}^{FM}\right) \in \mathcal{C}(X \times Y)$ in the flat limit. Hence the claim. ■

5.3 SYZ transforms commute with FM transforms

Recall from the last section that when $\mathcal{P}_{cx}^{FM}$ is the Poincaré sheaf for the FM transform between the elliptic CY manifold M and its dual elliptic CY manifold W, then its mirror $\mathcal{P}_{sym}^{FM}$ is the diagonal Lagrangian submanifold in $X \times Y$. Thus it defines a symplectic FM transform $\mathcal{F}_{sym}^{FM}$ from X to Y. We claim that the complex/symplectic Fourier-Mukai transforms commute with the SYZ transforms,

$$\mathcal{F}_{sym}^{FM} \circ \mathcal{F}^{SYZ} = \mathcal{F}^{SYZ} \circ \mathcal{F}_{cx}^{FM}.$$

as depicted in the following diagram.

$$
\begin{array}{ccc}
\begin{array}{c}\text{Complex}\\ \text{Geometry}\end{array} ((M)) & \overset{\mathcal{F}^{SYZ}}{\longleftrightarrow} & \begin{array}{c}\text{Symplectic}\\ \text{Geometry}\end{array} ((X)) \\
\mathcal{F}^{FM}_{cx} \updownarrow & & \updownarrow \mathcal{F}^{FM}_{sym} \\
\begin{array}{c}\text{Complex}\\ \text{Geometry}\end{array} ((W)) & \overset{\mathcal{F}^{SYZ}}{\longleftrightarrow} & \begin{array}{c}\text{Symplectic}\\ \text{Geometry}\end{array} ((Y))
\end{array}
$$

Since $\mathcal{F}^{FM}_{sym}$ is essentially an identity transformation, we can regard the FM transform for elliptic Calabi-Yau as a *square* of the SYZ transforms!

In fact our claim follows from a more general statement: If M and W are two Calabi-Yau manifolds, possibly of different dimensions and X and Y are their mirror manifolds respectively. Let $\mathcal{P}_{cx}$ be any complex cycle in $M \times W$ and $\mathcal{P}_{sym}$ be its mirror Lagrangian cycle in $X \times Y$. They define general complex/symplectic FM transforms $\mathcal{F}^{FM}_{cx}$ and $\mathcal{F}^{FM}_{sym}$ respectively. Then these FM transforms commute with the SYZ transforms.

The key point is the SYZ transform is an involution, i.e. $\mathcal{F}^{SYZ} \circ \mathcal{F}^{SYZ} = id$. Suppose $\mathcal{S}$ is a complex cycle in M, we want to show that

$$\mathcal{F}^{FM}_{sym}\left(\mathcal{F}^{SYZ}_{(M,X)}(\mathcal{S})\right) = \mathcal{F}^{SYZ}_{(W,X)}\left(\mathcal{F}^{FM}_{cx}(\mathcal{S})\right).$$

Equivalently,

$$(\pi_X)_*(\pi_M)_*\mathcal{S}_M \otimes \mathcal{P}^{SYZ}_{(M,X)} \otimes \mathcal{P}^{FM}_{sym} = (\pi_W)_*(\pi_M)_*\mathcal{S}_M \otimes \mathcal{P}^{FM}_{cx} \otimes \mathcal{P}^{SYZ}_{(W,Y)}.$$

Therefore, it suffices to prove that,

$$(\pi_X)_*\mathcal{P}^{SYZ}_{(M,X)} \otimes \mathcal{P}^{FM}_{sym} = (\pi_W)_*\mathcal{P}^{FM}_{cx} \otimes \mathcal{P}^{SYZ}_{(W,Y)},$$

over $M \times Y$. However,

$$
\begin{aligned}
\mathcal{P}^{FM}_{sym} &= (\pi_M)_*(\pi_W)_*\mathcal{P}^{FM}_{cx} \otimes \mathcal{P}^{SYZ}_{(M\times W, X\times Y)} \\
&= (\pi_M)_*(\pi_W)_*\mathcal{P}^{FM}_{cx} \otimes \mathcal{P}^{SYZ}_{(M,X)} \otimes \mathcal{P}^{SYZ}_{(W,Y)}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&(\pi_X)_*\mathcal{P}^{SYZ}_{(M,X)} \otimes \mathcal{P}^{FM}_{sym} \\
= \; &(\pi_X)_*\mathcal{P}^{SYZ}_{(M,X)} \otimes \left[(\pi_M)_*(\pi_W)_*\mathcal{P}^{FM}_{cx} \otimes \mathcal{P}^{SYZ}_{(M,X)} \otimes \mathcal{P}^{SYZ}_{(W,Y)}\right] \\
= \; &(\pi_W)_*\mathcal{P}^{FM}_{cx} \otimes \mathcal{P}^{SYZ}_{(W,Y)} \otimes \left[(\pi_M)_*(\pi_X)_*\mathcal{P}^{SYZ}_{(M,X)} \otimes \mathcal{P}^{SYZ}_{(M,X)}\right] \\
= \; &(\pi_W)_*\mathcal{P}^{FM}_{cx} \otimes \mathcal{P}^{SYZ}_{(W,Y)}.
\end{aligned}
$$

The last equality holds because SYZ transforms are involutive. Hence the result.
■

6 Conclusions and discussions

In this paper we have introduced the notion of a twin Lagrangian fibration and explained several of its properties. We argued via the SYZ proposal that the mirror manifold of an elliptic CY manifold should admit such a structure, possibly coupled with a non-trivial superpotential which we have not fully understood yet.

An important tool in the study of the complex geometry of elliptic manifolds is the FM transform. We argued that under the SYZ transform, this FM transform will become the identity transformation for the symplectic geometry between dual twin Lagrangian fibrations. Even though we are mostly interested in the elliptic fibration situation, these arguments can be applied to Abelian varieties fibrations, and possibly to general CY fibrations with suitable adjustments.

We could also study mirror of the symplectic geometry of the elliptic CY manifolds. For example let ω_M (resp. ω_S) be any Kähler form on M (resp. S). The pullback of ω_S under the elliptic fibration $M \to S$, denoted as ω_S again, is only nef but not ample. Obviously it satisfies $\omega_S^{n-1} \neq 0$ and $\omega_S^n = 0$. For any $t > 0$, $\omega_{M,t} = t\omega_M + (1-t)\,\omega_S$ is always a Kähler form on M. In particular it satisfies $\omega_{M,t}^n \neq 0$ and $\omega_{M,t}^{n+1} = 0$.

On the mirror side, these correspond to a two parameter family of complex structures on X where (i) the generic monodromy is of maximally unipotent and the corresponding vanishing cycles are the Lagrangian fibers of $X \to B$. (ii) a special monodromy T_0 for this family has the property $N_0^{n-1} \neq 0$ but $N_0^n = 0$ where $N_0 = \log(I - T_0)$ and the corresponding vanishing cycles are the $(n+1)$ dimensional coisotropic fibers of the composition map $X \to B \to D$.

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