A Uniform Description of Riemannian Symmetric Spaces as Grassmannians Using Magic Square

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Abstract

In this thesis we introduce and study the (i) Grassmannian, (ii) Lagrangian Grassmannian and (iii) double Lagrangian Grassmannian of subspaces in $(\mathbb{A} \otimes \mathbb{B})^n$, where $\mathbb{A}$ and $\mathbb{B}$ are normed division algebras, i.e. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.

We show that every irreducible compact Riemannian symmetric space X must be one of these Grassmannian spaces (up to a finite cover) or a compact simple Lie group. Furthermore, its noncompact dual symmetric space is the open submanifold of X consisting of spacelike linear subspaces, at least in the classical cases.

This gives a simple and uniform description of all symmetric spaces. This is analogous to Tits magic square description for simple Lie algebras.

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Contents

Chapter 1

Introduction

Riemannian symmetric spaces are important model spaces in geometry. They are classified in terms of symmetric Lie algebras. Every compact symmetric space has a noncompact dual and vice versa. Examples of compact symmetric spaces include *Grassmannians* of k-planes in n-dimensional vector spaces over $\mathbb{R}$, $\mathbb{C}$, or even $\mathbb{H}$. Symbolically, we write these symmetric spaces as $\mathrm{O}(n)/\mathrm{O}(k)\,\mathrm{O}(n-k) = \left\{\mathbb{R}^k \subset \mathbb{R}^n\right\}$, $\mathrm{U}(n)/\mathrm{U}(k)\,\mathrm{U}(n-k) = \left\{\mathbb{C}^k \subset \mathbb{C}^n\right\}$ and so on. There are also *Lagrangian Grassmannians* in $\mathbb{C}^n$ and $\mathbb{H}^n$, which we write as $\mathrm{U}(n)/\mathrm{O}(n) = \{\mathbb{R}^n \subset \mathbb{C}^n\}$ and $\mathrm{Sp}(n)/\mathrm{U}(n) = \{\mathbb{C}^n \subset \mathbb{H}^n\}$. Of course, compact Lie groups are also examples of compact symmetric spaces. For example $\mathrm{O}(n) = \mathrm{O}(n) \times \mathrm{O}(n)/\mathrm{O}(n)$ is the set of all maximally isotropic subspaces in $\mathbb{R}^n \oplus \mathbb{R}^n$ with respect to a symmetric 2-tensor g' of type (n, n). Taking g'-orthogonal complement defines an involution $\sigma_{g'}$ on the Grassmannian and we have $\mathrm{O}(n) = \{\mathbb{R}^n \subset \mathbb{R}^n \oplus \mathbb{R}^n\}^{\sigma_{g'}}$.

Classical compact Lie groups include $\mathrm{SO}(n)$, $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$. Roughly speaking, they are groups of isometries of $\mathbb{R}^n$, $\mathbb{C}^n$ and $\mathbb{H}^n$ respectively. There are also five exceptional Lie groups, namely G_2, F_4, E_6, E_7 and E_8. If we include the largest normed division algebra $\mathbb{O}$, the octonion, also known as the Cayley number Ca, then we can obtain G_2 and F_4 as well. Since $\mathbb{O}$ is not associative,

$\mathbb{O}^n$ does not make sense at all. However, if $n \leq 3$ then it is possible to define the group of symmetries of $\mathbb{O}^n$ using exceptional Jordan algebras (see e.g. [13]). For the remaining groups, Freudenthal [8] and Tits [15] introduced the *magic square* to describe *every* type of compact Lie algebras in a uniform manner by considering $(\mathbb{A} \otimes \mathbb{B})^n$ with both $\mathbb{A}$ and $\mathbb{B}$ normed division algebras. In particular, using the magic square, we can realize the Lie algebra of E_6, E_7 and E_8 as roughly the Lie algebras of infinitesimal isometries of $(\mathbb{C} \otimes \mathbb{O})^3$, $(\mathbb{H} \otimes \mathbb{O})^3$ and $(\mathbb{O} \otimes \mathbb{O})^3$ respectively.

1.1 Main Result – Compact Case

For compact Riemannian symmetric spaces, there are many more exceptional cases. At first sight, it seems hard to imagine that they can all be described in a simple and uniform manner. By realizing the symmetry of the magic square as a mirror duality between complex geometry and symplectic geometry, we generalize the magic square and succeed in giving a simple and uniform description of all compact Riemannian symmetric spaces. Among them, the most nontrivial ones are $E_6/\mathrm{Sp}(4)$ and $E_7/\mathrm{SU}(8)$, which will be described as the Grassmannians of *double Lagrangian subspaces* $\Lambda^2 (\mathbb{R} \otimes \mathbb{H})^4$ in $(\mathbb{C} \otimes \mathbb{O})^3$ and $\Lambda^2 (\mathbb{C} \otimes \mathbb{H})^4$ in $(\mathbb{H} \otimes \mathbb{O})^3$ respectively!

Main result - compact case: A description of four types of Grassmannian spaces which include every type of compact symmetric spaces.

First type: *Compact (semi)simple Lie groups*

$$G = \{(\mathbb{A} \otimes \mathbb{B})^n \subset (\mathbb{A} \otimes \mathbb{B})^n \oplus (\mathbb{A} \otimes \mathbb{B})^n\}^{\sigma_{g'}},$$

here $\sigma_{g'}$ is an involution induced by a canonical symmetric 2-tensor on $(\mathbb{A} \otimes \mathbb{B})^n \oplus (\mathbb{A} \otimes \mathbb{B})^n$. These spaces are given in the following table (with the Abelian part taken away). [1]

[1]For all tables in this paper, n is assumed to be 3 when $\mathbb{A}$ or $\mathbb{B}$ equals $\mathbb{O}$.

$\mathbb{A}\backslash\mathbb{B}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathrm{SO}(n)$	$\mathrm{SU}(n)$	$\mathrm{Sp}(n)$	F_4
$\mathbb{C}$	$\mathrm{SU}(n)$	$\mathrm{SU}(n)^2$	$\mathrm{SU}(2n)$	E_6
$\mathbb{H}$	$\mathrm{Sp}(n)$	$\mathrm{SU}(2n)$	$\mathrm{SO}(4n)$	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

(Table: C1)

On the Lie algebra level, it coincides with the magic square.

Recall that the compact Lie group G_2 is the automorphism group of $\mathbb{O}$, in fact it is of the first type for $n = 1$ and $\mathbb{A}$ is $\mathbb{O}$, $\mathbb{B}$ is $\mathbb{R}$, or vice versa, i.e. it can be realized as $\{\mathbb{O} \subset \mathbb{O} \oplus \mathbb{O}\}$ invariant under g'.

Second type: *Grassmannians*

$$Gr_{\mathbb{AB}}(k,n) = \left\{ (\mathbb{A} \otimes \mathbb{B})^k \subset (\mathbb{A} \otimes \mathbb{B})^n \right\}.$$

$\mathbb{A}\backslash\mathbb{B}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\frac{\mathrm{O}(n)}{\mathrm{O}(k)\mathrm{O}(n-k)}$	$\frac{\mathrm{U}(n)}{\mathrm{U}(k)\mathrm{U}(n-k)}$	$\frac{\mathrm{Sp}(n)}{\mathrm{Sp}(k)\mathrm{Sp}(n-k)}$	$\frac{F_4}{\mathrm{Spin}(9)}$
$\mathbb{C}$	$\frac{\mathrm{U}(n)}{\mathrm{U}(k)\mathrm{U}(n-k)}$	$\frac{\mathrm{U}(n)^2}{\mathrm{U}(k)^2\mathrm{U}(n-k)^2}$	$\frac{\mathrm{U}(2n)}{\mathrm{U}(2k)\mathrm{U}(2n-2k)}$	$\frac{E_6}{\mathrm{Spin}(10)\mathrm{U}(1)}$
$\mathbb{H}$	$\frac{\mathrm{Sp}(n)}{\mathrm{Sp}(k)\mathrm{Sp}(n-k)}$	$\frac{\mathrm{U}(2n)}{\mathrm{U}(2k)\mathrm{U}(2n-2k)}$	$\frac{\mathrm{O}(4n)}{\mathrm{O}(4k)\mathrm{O}(4n-4k)}$	$\frac{E_7}{\mathrm{Spin}(12)\mathrm{Sp}(1)}$
$\mathbb{O}$	$\frac{F_4}{\mathrm{Spin}(9)}$	$\frac{E_6}{\mathrm{Spin}(10)\mathrm{U}(1)}$	$\frac{E_7}{\mathrm{Spin}(12)\mathrm{Sp}(1)}$	$\frac{E_8}{\mathrm{SO}(16)}$

(Table: C2)

Third type: *Lagrangian Grassmannians* (with the Abelian part taken away)

$$LGr_{\mathbb{AB}}(n) = \left\{ \left(\frac{\mathbb{A}}{2} \otimes \mathbb{B}\right)^n \subset (\mathbb{A} \otimes \mathbb{B})^n \right\},$$

where $\frac{\mathbb{A}}{2}$ denotes $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ when $\mathbb{A}$ is $\mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ respectively.

$\mathbb{A}\backslash\mathbb{B}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{C}$	$\frac{\mathrm{SU}(n)}{\mathrm{SO}(n)}$	$\frac{\mathrm{SU}(n)^2}{\mathrm{SU}(n)}$	$\frac{\mathrm{SU}(2n)}{\mathrm{Sp}(n)}$	$\frac{E_6}{F_4}$
$\mathbb{H}$	$\frac{\mathrm{Sp}(n)}{\mathrm{U}(n)}$	$\frac{\mathrm{SU}(2n)}{S(\mathrm{U}(n)^2)}$	$\frac{\mathrm{SO}(4n)}{\mathrm{U}(2n)}$	$\frac{E_7}{E_6\mathrm{U}(1)}$
$\mathbb{O}$	$\frac{F_4}{\mathrm{Sp}(3)\mathrm{Sp}(1)}$	$\frac{E_6}{\mathrm{SU}(6)\mathrm{Sp}(1)}$	$\frac{E_7}{\mathrm{Spin}(12)\mathrm{Sp}(1)}$	$\frac{E_8}{E_7\mathrm{Sp}(1)}$

(Table: C3)

The compact symmetric space $G_2/\mathrm{Sp}\,(1)\,\mathrm{Sp}\,(1)$ is a Lagrangian Grassmannian for $n = 1$ and $\mathbb{A}$ is $\mathbb{O}$, $\mathbb{B}$ is $\mathbb{R}$, i.e. it can be realized as $\{\mathbb{H} \subset \mathbb{O}\}$.

Fourth type: *Double Lagrangian Grassmannians* (with the Abelian part taken away)

$$LLGr_{\mathbb{AB}} = \left\{ \left(\frac{\mathbb{A}}{2} \otimes \frac{\mathbb{B}}{2} \right)^n \oplus \left(j_1 \frac{\mathbb{A}}{2} \otimes j_2 \frac{\mathbb{B}}{2} \right)^n \subset (\mathbb{A} \otimes \mathbb{B})^3 \right\},$$

where j_1 and j_2 are elements of $\mathbb{A}$ and $\mathbb{B}$ respectively, such that $j_1^2 = j_2^2 = -1$ and $\mathbb{A} = \frac{\mathbb{A}}{2} + j_1 \frac{\mathbb{A}}{2}$ and $\mathbb{B} = \frac{\mathbb{B}}{2} + j_2 \frac{\mathbb{B}}{2}$.

$\mathbb{A}\backslash\mathbb{B}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{C}$	$\frac{\mathrm{SU}(n)^2}{\mathrm{SO}(n)^2}$	$\frac{\mathrm{SU}(2n)}{\mathrm{SO}(2n)}$	$\frac{E_6}{\mathrm{Sp}(4)}$
$\mathbb{H}$	$\frac{\mathrm{SU}(2n)}{\mathrm{SO}(2n)}$	$\frac{\mathrm{SO}(4n)}{\mathrm{S}(\mathrm{O}(2n)^2)}$	$\frac{E_7}{\mathrm{SU}(8)}$
$\mathbb{O}$	$\frac{E_6}{\mathrm{Sp}(4)}$	$\frac{E_7}{\mathrm{SU}(8)}$	$\frac{E_8}{\mathrm{SO}(16)}$

(Table: C4)

1.2 Main Result – Noncompact Case

As a corollary, we can also give a simple and uniform description of *all* Riemannian symmetric spaces of noncompact type. Recall that any noncompact symmetric space has a compact dual X. From the above main result, X parametrizes certain linear subspaces P in a fixed "vector space" $V \simeq (\mathbb{A} \otimes \mathbb{B})^n$. If we fix one such subspace P_0 and write V as an orthogonal decomposition $V = P_0 \oplus P_0^{\perp}$. We change the positive definite inner product $g_V = g_{P_0} \oplus g_{P_0^{\perp}}$ on V to an *indefinite* inner product $\check{g}_V = g_{P_0} \oplus \left(-g_{P_0^{\perp}}\right)$. Then the noncompact symmetric space $\check{X}$ consists precisely of those subspaces P in V which are of *spacelike*, i.e. $\check{g}_V(v,v) > 0$ for any nonzero vector v in P. Furthermore, this inclusion $\check{X} \subset X$ is an open embedding for at least any pair of dual symmetric spaces of classical types, i.e. the symmetric spaces whose isometry groups are classical groups.

Main result - noncompact case: A description of four types of spacelike Grassmannian spaces which include every type of noncompact symmetric spaces.

First type: *Noncompact dual to compact Lie groups* $G^{\mathbb{C}}/G$.

$\mathbb{A}\backslash\mathbb{B}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathrm{SO}(n,\mathbb{C})/\mathrm{SO}(n)$	$\mathrm{SL}(n,\mathbb{C})/\mathrm{SU}(n)$	$\mathrm{Sp}(2n,\mathbb{C})/\mathrm{Sp}(n)$	$F_4^{\mathbb{C}}/F_4$
$\mathbb{C}$	$\mathrm{SL}(n,\mathbb{C})/\mathrm{SU}(n)$	$\mathrm{SL}(n,\mathbb{C})^2/\mathrm{SU}(n)^2$	$\mathrm{SL}(2n,\mathbb{C})/\mathrm{SU}(2n)$	$E_6^{\mathbb{C}}/E_6$
$\mathbb{H}$	$\mathrm{Sp}(2n,\mathbb{C})/\mathrm{Sp}(n)$	$\mathrm{SL}(2n,\mathbb{C})/\mathrm{SU}(2n)$	$\mathrm{SO}(4n,\mathbb{C})/\mathrm{SO}(4n)$	$E_7^{\mathbb{C}}/E_7$
$\mathbb{O}$	$F_4^{\mathbb{C}}/F_4$	$E_6^{\mathbb{C}}/E_6$	$E_7^{\mathbb{C}}/E_7$	$E_8^{\mathbb{C}}/E_8$

(Table: N1)

Second type: *Spacelike Grassmannians* $Gr^{+}_{\mathbb{AB}}(k,n)$.

$\mathbb{A}\backslash\mathbb{B}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\frac{\mathrm{O}(k,n-k)}{\mathrm{O}(k)\mathrm{O}(n-k)}$	$\frac{\mathrm{U}(k,n-k)}{\mathrm{U}(k)\mathrm{U}(n-k)}$	$\frac{\mathrm{Sp}(k,n-k)}{\mathrm{Sp}(k)\mathrm{Sp}(n-k)}$	$\frac{F_{4(-20)}}{\mathrm{Spin}(9)}$
$\mathbb{C}$	$\frac{\mathrm{U}(k,n-k)}{\mathrm{U}(k)\mathrm{U}(n-k)}$	$\frac{\mathrm{U}(k,n-k)^2}{\mathrm{U}(k)^2\mathrm{U}(n-k)^2}$	$\frac{\mathrm{U}(2k,2n-2k)}{\mathrm{U}(2k)\mathrm{U}(2n-2k)}$	$\frac{E_{6(-14)}}{\mathrm{Spin}(10)\mathrm{U}(1)}$
$\mathbb{H}$	$\frac{\mathrm{Sp}(k,n-k)}{\mathrm{Sp}(k)\mathrm{Sp}(n-k)}$	$\frac{\mathrm{U}(2k,2n-2k)}{\mathrm{U}(2k)\mathrm{U}(2n-2k)}$	$\frac{\mathrm{O}(4k,4n-4k)}{\mathrm{O}(4k)\mathrm{O}(4n-4k)}$	$\frac{E_{7(-5)}}{\mathrm{Spin}(12)\mathrm{Sp}(1)}$
$\mathbb{O}$	$\frac{F_{4(-20)}}{\mathrm{Spin}(9)}$	$\frac{E_{6(-14)}}{\mathrm{Spin}(10)\mathrm{U}(1)}$	$\frac{E_{7(-5)}}{\mathrm{Spin}(12)\mathrm{Sp}(1)}$	$\frac{E_{8(8)}}{\mathrm{SO}(16)}$

(Table: N2)

Remark 1.2.1 *Here we use the same notation to denote the identity connected component of $O(k, n-k)/O(k)O(n-k)$, and similarly in the other cases.*

Where -20 in the group $F_{4(-20)}$ is the signature of the Killing form of the noncompact Lie group $F_{4(-20)}$, the other cases are similar.

Third type: *Spacelike Lagrangian Grassmannians* $LGr^{+}_{\mathbb{A}\mathbb{B}}(n)$ (with the Abelian part taken away).

$\mathbb{A}\backslash\mathbb{B}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{C}$	$\frac{\mathrm{SL}(n,\mathbb{R})}{\mathrm{SO}(n)}$	$\frac{\mathrm{SL}(n,\mathbb{C})}{\mathrm{SU}(n)}$	$\frac{\mathrm{SL}(n,\mathbb{H})}{\mathrm{Sp}(n)}$	$\frac{E_{6(-26)}}{F_4}$
$\mathbb{H}$	$\frac{\mathrm{Sp}(2n,\mathbb{R})}{\mathrm{U}(n)}$	$\frac{\mathrm{SU}(n,n)}{S(\mathrm{U}(n)^2)}$	$\frac{\mathrm{SO}^*(4n)}{\mathrm{U}(2n)}$	$\frac{E_{7(-25)}}{E_6\mathrm{U}(1)}$
$\mathbb{O}$	$\frac{F_{4(4)}}{\mathrm{Sp}(3)\mathrm{Sp}(1)}$	$\frac{E_{6(2)}}{\mathrm{SU}(6)\mathrm{Sp}(1)}$	$\frac{E_{7(-5)}}{\mathrm{Spin}(12)\mathrm{Sp}(1)}$	$\frac{E_{8(-24)}}{E_7\mathrm{Sp}(1)}$

(Table: N3)

Fourth type: *Spacelike Double Lagrangian Grassmannians* $LLGr^{+}_{\mathbb{A}\mathbb{B}}$ (with the Abelian part taken away).

$\mathbb{A}\backslash\mathbb{B}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{C}$	$\frac{\mathrm{SL}(n,\mathbb{R})^2}{\mathrm{SO}(n)^2}$	$\frac{\mathrm{SL}(2n,\mathbb{R})}{\mathrm{SO}(2n)}$	$\frac{E_{6(6)}}{\mathrm{Sp}(4)}$
$\mathbb{H}$	$\frac{\mathrm{SL}(2n,\mathbb{R})}{\mathrm{SO}(2n)}$	$\frac{\mathrm{SO}(2n,2n)}{\mathrm{S}(\mathrm{O}(2n)^2)}$	$\frac{E_{7(7)}}{\mathrm{SU}(8)}$
$\mathbb{O}$	$\frac{E_{6(6)}}{\mathrm{Sp}(4)}$	$\frac{E_{7(7)}}{\mathrm{SU}(8)}$	$\frac{E_{8(8)}}{\mathrm{SO}(16)}$

(Table: N4)

Remark 1.2.2 *A Riemannian symmetric space G/K admits a G-invariant complex structure precisely when K has a $U(1)$-factor. Such spaces are called Hermitian symmetric spaces and they play very important roles in many different branches of mathematics. From the above tables we know that Hermitian symmetric spaces are precisely those grassmannians parametrizing certains types of*

$(\mathbb{A}\otimes\mathbb{B})^k$ *in* $W=(\mathbb{D}\otimes\mathbb{E})^n$ *where* $\mathbb{A}$ *or* $\mathbb{B}$ *equals the complex numbers* $\mathbb{C}$*, with only one exception* $O(2,n)/O(2)O(n)$.

In the remainder of this section, we give an intuitive explanation of the origin of these four types of Grassmannians, we first recall that all normed division algebras $\mathbb{A}$ can be obtained from a *doubling* construction starting from $\mathbb{R}$, the Cayley-Dickson process: $\mathbb{R}\rightsquigarrow\mathbb{C}\rightsquigarrow\mathbb{H}\rightsquigarrow\mathbb{O}$. For any "vector space" V over $\mathbb{R}$, this doubling procedure can be viewed in two different ways: (i) complexification $V\longmapsto(TV,J)$ and (ii) symplectification $V\longmapsto(T^*V,\omega)$. The corresponding semi-simple part of Lie algebras of infinitesimal symmetries of these doublings are (i) $\mathfrak{sl}(n,\mathbb{R})\longmapsto\mathfrak{sl}(n,\mathbb{C})\longmapsto\mathfrak{sl}(n,\mathbb{H})$ and (ii) $\mathfrak{sl}(n,\mathbb{R})\longmapsto\mathfrak{sp}(2n,\mathbb{R})\longmapsto\mathfrak{sp}(2n,\mathbb{C})$. The octonion case is more complicated which we will deal with in the next section via the magic square. Combining both of them gives the following table of Lie algebras.

$\mathbb{A}\backslash\mathbb{B}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathbb{R}$	$\mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{H})$
$\mathbb{C}$	$\mathfrak{sp}(2n,\mathbb{R})$	$\mathfrak{su}(n,n)$	$\mathfrak{so}^*(4n)$
$\mathbb{H}$	$\mathfrak{sp}(2n,\mathbb{C})$	$\mathfrak{sl}(2n,\mathbb{C})$	$\mathfrak{so}(4n,\mathbb{C})$

(Table: T1)

If we also require these symmetries to preserve the metric g on V, then they reduce to the maximal compact (modulo center) subalgebras and we obtain the magic square

$\mathbb{A}\backslash\mathbb{B}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathbb{R}$	$\mathfrak{so}(n)$	$\mathfrak{su}(n)$	$\mathfrak{sp}(n)$
$\mathbb{C}$	$\mathfrak{su}(n)$	$\mathfrak{su}(n)^2$	$\mathfrak{su}(2n)$
$\mathbb{H}$	$\mathfrak{sp}(n)$	$\mathfrak{su}(2n)$	$\mathfrak{so}(4n)$

(Table: T2)

The symmetry of the magic square along the diagonal is closely related to the formula

$$g\left(Ju,v\right)=\omega\left(u,v\right),$$

which says that complex structure and symplectic structure determine each other when a matric is given.

Notice that every symplectic form ω on V determines an involution σ_ω on the Grassmannian of linear subspaces $P\subset V$ by $\sigma_\omega\left(P\right)=P^{\perp_\omega}$ whose fix points are precisely Lagrangian subspaces in (V,ω). This is analogous to every complex structure J on V determines an involution σ_J on the Grassmannian by $\sigma_J\left(P\right)=JP$ whose fix points are precisely complex linear subspaces in (V,J).

Suppose that $V\simeq\left(\mathbb{A}\otimes\mathbb{B}\right)^n$ with $\mathbb{A}$, $\mathbb{B}$ normed division algebras. Then the Grassmannian of all real linear subspaces P in V which are σ_J-invariant for every J coming from either $\mathbb{A}$ or $\mathbb{B}$ is precisely our compact symmetric space of type I, $Gr_{\mathbb{AB}}\left(k,n\right)$. Similarly, if we require P to be σ_ω-invariant, instead of σ_J-invariant, for one J coming from $\mathbb{A}$, then we obtain the Lagrangian Grassmannians $LGr_{\mathbb{AB}}\left(n\right)$. Furthermore, if we do this to both $\mathbb{A}$ and $\mathbb{B}$, then we obtain the double Lagrangian Grassmannian $LLGr_{\mathbb{AB}}(n)$.

Recall that the canonical symplectic form ω on $T^*V=V\oplus V^*$ is the skew-symmetric component of a two tensor induced from the natural pairing between V and V^*. The symmetric component of this tensor is an indefinite inner product h on $V\oplus V^*$ with signature (N,N), where N is the dimension of V over $\mathbb{R}$. If we replace σ_ω by σ_h in the definition of $LGr_{\mathbb{AB}}\left(n\right)$, then we obtain the list of compact Lie groups G.

In conclusion, every irreducible compact symmetric space is a Grassmannian of linear subspaces in $W\simeq\left(\mathbb{A}\otimes\mathbb{B}\right)^n$ which are invariant under involutions σ_J, σ_ω or σ_h.

Chapter 2

Magic Square from Complex and Symplectic Point of Views

Complex simple Lie algebras are completely classified. The classical ones are complexification of $\mathfrak{so}(n)$, $\mathfrak{su}(n)$ and $\mathfrak{sp}(n)$, which are Lie algebras of isometry groups (modulo center) of $\mathbb{R}^n$, $\mathbb{C}^n$ and $\mathbb{H}^n$ respectively. They can also be identified as derivation algebras of Jordan algebras of rank n Hermitian matrices over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ respectively. Namely $\mathfrak{so}(n) = \mathrm{Der}H_n(\mathbb{R})$, $\mathfrak{su}(n) = \mathrm{Der}H_n(\mathbb{C})$ and $\mathfrak{sp}(n) = \mathrm{Der}H_n(\mathbb{H})$. The rest are five exceptional Lie algebras, $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$ and $\mathfrak{e}_8$. To define them, we need to include the non-associative normed division algebra $\mathbb{O}$, the octonion or the Cayley number. For instance $\mathfrak{g}_2 = \mathrm{Der}\mathbb{O}$ and $\mathfrak{f}_4 = \mathrm{Der}H_3(\mathbb{O})$. Even though $\mathbb{O}^n$ can not be defined properly due to the non-associative nature of $\mathbb{O}$, nevertheless, the Jordan algebra $H_n(\mathbb{O})$ has a natural interpretation as the exceptional Jordan algebra when $n = 3$ (see e.g. [10]).

Tits [15] observed that we can have a uniform description of all simple Lie algebras, including $\mathfrak{e}_6$, $\mathfrak{e}_7$ and $\mathfrak{e}_8$, if we use any two normed division algebras $\mathbb{A}$, $\mathbb{B} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and define

$$L_3(\mathbb{A}, \mathbb{B}) = \mathrm{Der}H_3(\mathbb{A}) \oplus (H_3'(\mathbb{A}) \otimes \mathrm{Im}\,\mathbb{B}) \oplus \mathrm{Der}(\mathbb{B}),$$

where $H_3'(\mathbb{A})$ is the subset of trace zero Hermitian matrices in $H_3(\mathbb{A})$.

2.1 Magic Square

Let $\mathbb{K}$ be an algebra over $\mathbb{R}$ with a quadratic form $x \mapsto |x|^2$ and associated bilinear form $\langle x, y\rangle$. If the quadratic form satisfies

$$|xy|^2 = |x|^2|y|^2, \quad \forall x, y \in \mathbb{K},$$

then $\mathbb{K}$ is a composition algebra. A division algebra is an algebra in which

$$xy = 0 \ \Rightarrow \ x = 0 \text{ or } y = 0.$$

This is true in a composition algebra if the quadratic form $|x|^2$ is positive definite. By Hurwitz's theorem [12], the only such algebras are $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$. The positive definite quadratic form is also called a norm on $\mathbb{K}$, hence $\mathbb{K}$ is also called a normed division algebra. These algebras can be obtained by the Cayley-Dickson process [12]. We consider $\mathbb{R}$ to be embedded in $\mathbb{K}$ as the set of scalar multiples of the identity element, and denote by $\mathrm{Im}\mathbb{K}$ the subspace of $\mathbb{K}$ orthogonal to $\mathbb{R}$, so that $\mathbb{K} = \mathbb{R} \oplus \mathrm{Im}\mathbb{K}$. We write $x = \mathrm{Re}x + \mathrm{Im}x$ with $\mathrm{Re}x \in \mathbb{R}$ and $\mathrm{Im}x \in \mathrm{Im}\mathbb{K}$. The conjugation of x is $\overline{x} = \mathrm{Re}x - \mathrm{Im}x$, satisfies

$$\overline{xy} = \overline{y}\ \overline{x},$$

and

$$x\overline{x} = |x|^2.$$

The inner product in $\mathbb{K}$ is given in terms of this conjugation as

$$\langle x, y\rangle = \mathrm{Re}(x\overline{y}) = \mathrm{Re}(\overline{x}y).$$

Let $M_n(\mathbb{K})$ be the algebra of all $n \times n$ matrices with entries in $\mathbb{K}$, where $\mathbb{K}$ is a normed division algebra, i.e. $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$.

For $A \in M_n(\mathbb{K})$, let A^t and $\overline{A}$ denote the transpose and conjugate of A respectively, defined as

$$(A^t)_{ij} = A_{ji}$$

and

$$(\overline{A})_{ij} = \overline{A_{ij}}.$$

By using these notations, the hermitian conjugate of the matrix A is $\overline{A}^t$.

Let I denote the identity matrix (of a size which will be clear from the context). We use $\mathrm{U}(s,t)$ ($\mathrm{SU}(s,t)$) for the pseudo-unitary (unimodular pseudo-unitary) group,

$$\mathrm{U}(s,t) = \left\{ A \in M_n(\mathbb{C}) : AG\overline{A}^t = G \right\}$$

where $G = \mathrm{diag}(1, ..., 1, -1, ..., -1)$ with s $+$ signs and t $-$ signs; $\mathrm{Sp}(n)$ for the group of antihermitian quaternionic matrices A,

$$\mathrm{Sp}(n) = \left\{ A \in M_n(\mathbb{H}) : A\overline{A}^t = I \right\};$$

and $\mathrm{Sp}(2n, \mathbb{K})$ for the symplectic group of $2n \times 2n$ matrices with entries in $\mathbb{K}$, i.e.

$$\mathrm{Sp}(2n, \mathbb{K}) = \left\{ A \in M_n(\mathbb{K}) : A\omega A^t = \omega \right\}$$

where $\omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. We also have $\mathrm{O}(s,t)$, the pseudo-orthogonal group, given by

$$\mathrm{O}(s,t) = \left\{ AGA^t = G \right\}$$

where G is defined as before. We will denote the Lie algebras of $\mathrm{SU}(s,t)$, $\mathrm{Sp}(n)$, $\mathrm{Sp}(2n, \mathbb{K})$ and $\mathrm{O}(s,t)$ as $\mathfrak{su}(s,t)$, $\mathfrak{sp}(n)$, $\mathfrak{sp}(2n, \mathbb{K})$ and $\mathfrak{so}(s,t)$ respectively. When s or t is zero, the groups $\mathrm{U}(s,t)$ ($\mathrm{SU}(s,t)$) and $\mathrm{O}(s,t)$ become the normal unitary and orthogonal groups $\mathrm{U}(n)$ ($\mathrm{SU}(n)$) and $\mathrm{O}(n)$ respectively.

A Jordan algebra $\mathbb{J}$ is defined to be a commutative algebra (over a field $\mathbb{K}$) in which all products satisfy the Jordan identity

$$(xy)x^2 = x(yx^2).$$

Let $H_n(\mathbb{K})$ and $A_n(\mathbb{K})$ be the sets of all hermitian and antihermitian matrices with entries in $\mathbb{K}$ respectively. We denote by $H'_n(\mathbb{K})$, $A'_n(\mathbb{K})$ and $M'_n(\mathbb{K})$ the subspaces of traceless matrices of $H_n(\mathbb{K})$, $A_n(\mathbb{K})$ and $M_n(\mathbb{K})$ respectively. If we define

$$X \cdot Y = XY + YX,$$

then $H_n(\mathbb{K})$ is a Jordan algebra for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ for all n and for $\mathbb{K} = \mathbb{O}$ when $n = 2, 3$.

For any algebra A, we define the derivation algebra $\mathrm{Der}(A)$ as

$$\mathrm{Der}(A) = \{D \mid D(xy) = D(x)y + xD(y) \ \forall x, y \in A\}.$$

The derivation algebras of the four normed division algebras are as follows:

$$\begin{aligned} &\mathrm{Der}(\mathbb{R}) = \mathrm{Der}(\mathbb{C}) = 0, \\ &\mathrm{Der}(\mathbb{H}) = \{C_a \mid a \in \mathrm{Im}\mathbb{H}, C_a(q) = aq - qa\} \cong \mathfrak{sp}(1), \\ &\mathrm{Der}(\mathbb{O}) \cong \mathfrak{g}_2, \end{aligned}$$

where $\mathfrak{g}_2$ is the compact exceptional Lie algebra of type G_2.

2.1.1 Magic square of 3×3 matrices

The following is the Tits original constructions of the magic square.

Let $\mathbb{A}$ and $\mathbb{B}$ be normed division algebras, define

$$L_3(\mathbb{A}, \mathbb{B}) = \mathrm{Der}H_3(\mathbb{A}) \oplus H_3'(\mathbb{A}) \otimes \mathrm{Im}\mathbb{B} \oplus \mathrm{Der}(\mathbb{B}).$$

This is a Lie algebra with Lie subalgebras $\mathrm{Der}H_3(\mathbb{A})$ and $\mathrm{Der}(\mathbb{B})$ when taken with the brackets

$$\begin{aligned} [D, A \otimes x] &= D(A) \otimes x \\ [E, A \otimes x] &= A \otimes E(x) \\ [D, E] &= 0 \\ [A \otimes x, B \otimes y] &= \frac{1}{6}\langle A, B\rangle D_{x,y} + (A * B) \otimes \frac{1}{2}[x, y] - \langle x, y\rangle [L_A, L_B] \end{aligned}$$

with $D \in \mathrm{Der}H_3(\mathbb{A})$; $A, B \in H_3'(\mathbb{A})$; $x, y \in \mathrm{Im}\mathbb{B}$ and $E \in \mathrm{Der}(\mathbb{B})$. These brackets are obtained from Schafer's description of the Tits construction [12]. We explain these operations as follows. $\langle A, B\rangle$ and $\langle x, y\rangle$ denote the symmetric bilinear forms on $H_3(\mathbb{A})$ and $\mathbb{B}$ respectively, given by

$$\langle A, B\rangle = \mathrm{Re}(\mathrm{tr}(A \cdot B)) = 2\mathrm{Re}(\mathrm{tr}(AB)),$$
$$\langle x, y\rangle = \frac{1}{2}(|x+y|^2 - |x|^2 - |y|^2) = \mathrm{Re}(x\overline{y}).$$

The derivation $D_{x,y}$ is defined as

$$D_{x,y} = [L_x, L_y] + [L_x, R_y] + [R_x, R_y] \ \in \mathrm{Der}(\mathbb{B}),$$

where L_x, R_x are maps of left and right multiplication by x respectively. Finally $(A * B)$ is the traceless part of the Jordan product of A and B,

$$A * B = A \cdot B - \frac{1}{3}\mathrm{tr}(A \cdot B).$$

Tits [15] showed that this gives a unified construction leading to the so-called magic square of Lie algebras of 3×3 matrices as follows.

$\mathbb{A}\backslash\mathbb{B}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sp}(3)$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{su}(3)$	$\mathfrak{su}(3)^2$	$\mathfrak{su}(6)$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{sp}(3)$	$\mathfrak{su}(6)$	$\mathfrak{so}(12)$	$\mathfrak{e}_6$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

Remark 2.1.1 *There are several versions of the magic square:*

1. *The Vinberg's version [16]*

$$L_3(\mathbb{A}, \mathbb{B}) = Der(\mathbb{A}) \oplus Der(\mathbb{B}) \oplus A_3'(\mathbb{A} \otimes \mathbb{B}),$$

where $A_n'(\mathbb{A} \otimes \mathbb{B})$ is the trace zero skew hermitian matrix of rank n with entries in $\mathbb{A} \otimes \mathbb{B}$;

2. The Barton-Sudbery version [4]

$$L_3(\mathbb{A},\mathbb{B}) = \mathfrak{tri}(\mathbb{A}) \oplus \mathfrak{tri}(\mathbb{B}) \oplus (\mathbb{A}\otimes\mathbb{B})^3,$$

where $\mathfrak{tri}(\mathbb{A})$ *is the triality algebra of* $\mathbb{A}$*, which by definition is a lie subalgebra of* $\mathfrak{so}(\mathbb{A}) \oplus \mathfrak{so}(\mathbb{A}) \oplus \mathfrak{so}(\mathbb{A})$*, satisfying*

$$A(xy) = x(By) + (Cx)y,\ \forall x, y \in \mathbb{A},$$

for any $(A, B, C) \in \mathfrak{tri}(\mathbb{A})$*, where* $\mathfrak{so}(\mathbb{A})$ *is the Lie algebra of the unimodular orthogonal group* $SO(\mathbb{A})$*, where* $\mathbb{A}$ *viewed as a real vector space.*

2.1.2 Magic square of 2×2 matrices

The Tits construction can also be adapted for 2×2 matrix algebras. In this case the underlying vector space is

$$L_2(\mathbb{A},\mathbb{B}) = \mathrm{Der}H_2(\mathbb{A}) \oplus H_2'(\mathbb{A}) \otimes \mathrm{Im}\mathbb{B} \oplus \mathfrak{so}(\mathrm{Im}\mathbb{B}),$$

which admits a Lie algebra structure with the Lie brackets given as follows,

$$\begin{aligned}
[D, A\otimes x] &= D(A)\otimes x \\
[E, A\otimes x] &= A\otimes E(x) \\
[D, E] &= 0 \\
[A\otimes x, B\otimes y] &= \frac{1}{4}\langle A, B\rangle D_{x,y} - \langle x, y\rangle [R_A, R_B]
\end{aligned}$$

where the symbols used in this set of brackets are defined in the same way as the ones used in the 3×3 case. This gives the compact magic square for 2×2 matrix algebras

$$L_2(\mathbb{A},\mathbb{B}) = \mathfrak{so}(\nu_1+\nu_2),$$

where ν_1, ν_2 are the dimensions of $\mathbb{A}, \mathbb{B}$ over $\mathbb{R}$. The magic square is as follows

$\mathbb{A}\backslash\mathbb{B}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}(2)$	$\mathfrak{so}(3)$	$\mathfrak{so}(5)$	$\mathfrak{so}(9)$
$\mathbb{C}$	$\mathfrak{so}(3)$	$\mathfrak{so}(4)$	$\mathfrak{so}(6)$	$\mathfrak{so}(10)$
$\mathbb{H}$	$\mathfrak{so}(5)$	$\mathfrak{so}(6)$	$\mathfrak{so}(8)$	$\mathfrak{so}(12)$
$\mathbb{O}$	$\mathfrak{so}(9)$	$\mathfrak{so}(10)$	$\mathfrak{so}(12)$	$\mathfrak{so}(16)$

Remark 2.1.2 *As a matter of fact, $L_n(\mathbb{A}, \mathbb{B})$ can be defined for any n as long as $\mathbb{A}$, $\mathbb{B} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. The following is the table of magic square:*

$\mathbb{A}\backslash\mathbb{B}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathbb{R}$	$\mathfrak{so}(n)$	$\mathfrak{su}(n)$	$\mathfrak{sp}(n)$
$\mathbb{C}$	$\mathfrak{su}(n)$	$\mathfrak{su}(n)^2$	$\mathfrak{su}(2n)$
$\mathbb{H}$	$\mathfrak{sp}(n)$	$\mathfrak{su}(2n)$	$\mathfrak{so}(4n)$

Freudenthal [8] also found a very different version to construct the magic square in about 1958.

2.1.3 Symmetry of the magic square

A surprising fact of the magic square is the symmetry in $\mathbb{A}$ and $\mathbb{B}$. For example $L_n(\mathbb{R}, \mathbb{R}) = \mathrm{Der} H_n(\mathbb{R}) = \mathfrak{so}\,(n)$ is the space of infinitesimal isometries of $V \cong \mathbb{R}^n$. $L_n(\mathbb{C}, \mathbb{R}) = \mathrm{Der} H_n(\mathbb{C}) = \mathfrak{su}\,(n)$ is the space of infinitesimal isometries of $V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus V = TV$ which also preserve the natural complex structure J on TV, the tangent bundle of V. On the other hand, any symmetric matrix $A = (A_{ij}) \in H_n'(\mathbb{R}) \subseteq L_n(\mathbb{R}, \mathbb{C})$ induces a natural transformation ϕ_A on $V \oplus V^* = T^*V$ preserving the canonical symplectic structure on the cotangent bundle,

$$\omega = \textstyle\sum_{i=1}^n dx^i \wedge dy_i,$$

where $\{x^i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ are dual coordinates on V and V^* respectively. Here $\phi_A(x^i) = x^i$ and $\phi_A(y_i) = y_i + \Sigma_j A_{ij} x^j$. Using the metric to identify V and V^*, the complex and symplectic structures on $TV = T^*V$ are related by the formula

$\omega(u,v) = g(Ju,v)$. Thus we obtain the first symmetry $L_n(\mathbb{C},\mathbb{R}) = L_n(\mathbb{R},\mathbb{C})$ in the magic square. By repeating this doubling process, we obtain other symmetries in the magic square.

We will also define noncompact Lie algebra $\mathfrak{n}_n(\mathbb{A},\mathbb{B})$, which contains $L_n(\mathbb{A},\mathbb{B})$ as its maximal compact Lie subalgebra (modulo center), by not requiring its elements to preserve the inner product. We will discuss this noncompact magic square and its properties in section 2.3.

In the next section, we will apply this construction to study the classification of symmetric spaces, we will identify any irreducible symmetric spaces as all (multi-complex) Grassmannian-Lagrangian linear cycles in the constructed spaces.

2.2 Doubling Construction

Definition 2.2.1 *Given any real vector space V, we define inductively $V^{[m,l]}$ by (i) $V^{[0,0]} = V$ and (ii) $V^{[m,l]} = TV^{[m,l-1]} = T^*V^{[m-1,l]}$, together with the l canonical complex structures $\left\{J_1^R, ..., J_l^R\right\}$ and m canonical symplectic structures $\left\{\omega_1^L, ..., \omega_m^L\right\}$ defined as below.*

To explain the complex and symplectic structures we assume that V is a $2n$ dimensional real vector space together with a complex structure and a symplectic structure as follows,

$$
\begin{aligned}
J_1 &= \sum(\partial_{x^i} \otimes dx^{n+i} - \partial_{x^{n+i}} \otimes dx^i), \\
\omega_1 &= \sum(dx^i \wedge dx^{n+i}),
\end{aligned}
$$

where $\{x^i\}_{i=1}^{2n}$ are the real coordinates of V, $\partial_{x^i} = \frac{\partial}{\partial x^i}$.

1. On TV, besides the canonical complex structure J_2, we also have a natural complex structure and a symplectic structure induced by J_1 and ω_1 given as follows, which we still use the same notations.

$$
\begin{aligned}
J_1 &= \sum(\partial_{x^i} \otimes dx^{n+i} - \partial_{x^{n+i}} \otimes dx^i) - \sum(\partial_{y_i} \otimes dy_{n+i} - \partial_{y_{n+i}} \otimes dy_i), \\
J_2 &= \sum(\partial_{x^i} \otimes dy^i - \partial_{y^i} \otimes dx^i), \\
\omega_1 &= \sum(dx^i \wedge dx^{n+i} + dy^i \wedge dy^{n+i}),
\end{aligned}
$$

   where $\{y^i\}_{i=1}^{2n}$ are the fiber coordinates of $\sum y^i \partial_{x^i}$ on the tangent bundle.

2. On T^*V, besides the canonical symplectic structure ω_2, we also have a natural complex structure and a symplectic structure induced by J_1 and ω_1 given as follows, which we still use the same notations.

$$
\begin{aligned}
J_1 &= \sum(\partial_{x^i} \otimes dx^{n+i} - \partial_{x^{n+i}} \otimes dx^i) + \sum(\partial_{u_i} \otimes du_{n+i} - \partial_{u_{n+i}} \otimes du_i), \\
\omega_1 &= \sum(dx^i \wedge dx^{n+i} - du_i \wedge du_{n+i}), \\
\omega_2 &= \sum(dx^i \wedge du_{n+i}),
\end{aligned}
$$

where $\{u_i\}_{i=1}^{2n}$ are the fiber coordinates of $\sum u_i dx^i$ on the cotangent bundle.

Proposition 2.2.1 *Above complex and symplectic structures on TV and T^*V satisfy the following conditions.*

$$J_1 J_2 = -J_2 J_1,$$

$$\omega_i(J_j(\cdot), J_j(\cdot)) = \omega_i(\cdot, \cdot),$$

$$\iota_{\omega_1}^{-1} \circ \iota_{\omega_2} = -\iota_{\omega_2}^{-1} \circ \iota_{\omega_1}.$$

Here $\iota_\omega : V \to V^*$ is defined as $\iota_\omega(v) = v \lrcorner\, \omega = \omega(v, \cdot)$.

This proposition can be easily proven by simple calculations.

2.3 Noncompact Magic Square

Given a vector space $V = V^{[0,0]} \cong \mathbb{R}^n$, its group of automorphisms is $\mathrm{GL}(n, \mathbb{R})$. Recall that $V^{[0,1]}$ (resp. $V^{[1,0]}$) carries a natural complex (resp. symplectic) structure and its group of automorphisms is $\mathrm{GL}(n, \mathbb{C})$ (resp. $\mathrm{Sp}(2n, \mathbb{R})$). If we restrict our attentions to only isometries, then both $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{Sp}(2n, \mathbb{R})$ reduce to $\mathrm{U}(n)$, which is also their common maximal compact subgroup. If we only look at semi-simple parts of these reductive groups, then they become $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{Sp}(2n, \mathbb{R})$ and $\mathrm{SU}(n)$ respectively. The following definitions generalize these to a general $V^{[m,l]}$.

Definition 2.3.1 *Given a real vector space $V \cong \mathbb{R}^n$, we denote by $N_n(m, l) \subset GL\left(n \cdot 2^{m+l}, \mathbb{R}\right)$ the connected maximal semisimple subgroup of the group of all automorphisms of $V^{[m,l]}$ which preserve the canonical complex structures $J_1^R, ..., J_l^R$ and symplectic structures $\omega_1^L, ..., \omega_m^L$ on $V^{[m,l]}$. We denote the maximal compact subgroup of $N_n(m, l)$ as $G_n(m, l)$, and the maximal semisimple subgroup of $G_n(m, l)$ as $G_n^s(m, l)$.*

The Lie algebras of $N_n(m, l)$ and $G_n(m, l)$ are denoted as $\mathfrak{n}_n(m, l)$ and $\mathfrak{g}_n(m, l)$ respectively.

Recall the Tits definition for $L_n(\mathbb{A}, \mathbb{B})$, in fact $L_n(\mathbb{A}, \mathbb{B})$ is just the Lie algebra of $G_n^s(m, l)$, where $\mathbb{A}$ and $\mathbb{B}$ are division algebras of dimension 2^m and 2^l respectively.

The main result in this section is the following theorem.

Theorem 2.3.2 *The groups $N_n(m, l)$, $G_n(m, l)$ and $G_n^s(m, l)$ satisfy the following properties:*

(i) $N_n(0, l)/G_n(0, l)$ is the noncompact dual to $G_n^s(1, l)/G_n(0, l)$.

(ii) $N_n(2, l)$ is the complexification of $N_n(1, l)$.

(iii) $N_n(1, l)/G_n(1, l)$ is the noncompact dual to $G_n^s(2, l)/G_n(1, l)$.

If we restrict our attention to $0 \leq m, l \leq 2$, then the Lie algebra of $G_n^s(m,l)$ is the same as the corresponding Lie algebra $L_n(\mathbb{A},\mathbb{B})$ in the Tits magic square where $dim_{\mathbb{R}}\mathbb{A} = 2^m$ and $dim_{\mathbb{R}}\mathbb{B} = 2^l$. Then we are naturally led to extend our definitions to $N_3(m,3)$, using (i), (ii) and (iii) in theorem 2.3.2. Thus we have the following table for $N_n(m,l)$.

$m\backslash l$	0	1	2	3 $(n=3)$
0	$SL(n,\mathbb{R})$	$SL(n,\mathbb{C})$	$SL(n,\mathbb{H})$	$E_{6(-26)}$
1	$Sp(2n,\mathbb{R})$	$SU(n,n)$	$SO^*(4n)$	$E_{7(-25)}$
2	$Sp(2n,\mathbb{C})$	$SL(2n,\mathbb{C})$	$SO(4n,\mathbb{C})$	${E_7}^{\mathbb{C}}$

Recall that suppose A is a linear transformation of a vector space V. We say (i) A preserves a complex structure J on V if $A \circ J = J \circ A$ and (ii) A preserves a symplectic structure ω on V if $\omega(Au, Av) = \omega(u,v)$ for any u, v in V. The following lemma expresses this condition in terms of the isomorphism ι_ω.

Lemma 2.3.1 *Suppose (V,ω) is a symplectic vector space and $A \in End(V)$. Then A preserves ω if and only if $\iota_\omega \circ A = A^* \circ \iota_\omega$, where $A^* \in End(V^*)$ is the adjoint of A.*

Proof. Replacing v by $A^{-1}v$, A preserves ω implies

$$\omega(Au, v) = \omega(u, A^{-1}v).$$

For any $u, v \in V$,

$$\begin{aligned}((\iota_\omega \circ A)u)(v) &= \omega(Au, v),\\ ((A^* \circ \iota_\omega)u)(v) &= \iota_\omega(u)(A^{-1}v)\\ &= \omega(u, A^{-1}v),\end{aligned}$$

so

$$\iota_\omega \circ A = A^* \circ \iota_\omega.$$

is equivalent to A preserves ω. □

Now we consider the group $N_n(m,l)$ for $m,l = 0,1,2$.

2.3.1 The case of $m = 0$

The case for $m = 0$ is obvious $\mathrm{SL}(n,\mathbb{K})$, where $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ when $l = 0,1,2$ respectively. The reason is that we can view $TV = V^{[0,1]}$ and $T(TV) = V^{[0,2]}$ together with the canonical complex structures as $\mathbb{C}^n$ and $\mathbb{H}^n$ respectively since the two complex structures on $T(TV)$ are anti-commutative:

$$J_1^R J_2^R = -J_2^R J_1^R.$$

Before we consider the other cases we first identify $V^{[m,l]}$ together with m symplectic structures and l complex structures as $(T^*)^m\mathbb{K}^n$ with m symplectic structures, where $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ when $l = 0,1,2$ respectively. Then we view any $A \in N_n(m,l)$ as $\mathbb{K}$-linear transformation $A \in \mathrm{SL}(n,\mathbb{K})$ preserving ω_i^L. In terms of matrices with entries in $\mathbb{K}$, this means

$$A\omega_i^L\,\overline{A}^t = \omega_i^L,$$

which deduced from

$$\omega_i^L(J_j^R(\cdot), J_j^R(\cdot)) = \omega_i(\cdot,\cdot).$$

At the Lie algebra level it is

$$A\omega_i^L + \omega_i^L\,\overline{A}^t = 0,$$

where $A \in \mathfrak{sl}(n,\mathbb{K})$. From now on we identify $N_n(m,l)$ as the maximal semisimple subgroup of all $\mathbb{K}$-linear transformations of $(T^*)^m\mathbb{K}^n$ which preserve ω_i^L ($i = 1,...,m$).

2.3.2 The case of $m = 1, l = 0$

The group $N_n(1,0)$ is the maximal semisimple subgroup of the group of all $\mathbb{R}$-linear transformations of $\mathbb{R}^{2n}$ which preserve the symplectic structure ω_1^L, so this is the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$.

2.3.3 The case of $m = l = 1$

Denote C the matrix $\frac{\sqrt{2}}{2}\begin{pmatrix} iI & I \\ I & iI \end{pmatrix}$, where $i = \sqrt{-1}$, we have

$$C\,\overline{C}^t = I$$

and

$$C\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\overline{C}^t = \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix}.$$

For any $A \in N_n(1,1)$, we have

$$\begin{aligned}
&\left(CA\overline{C}^t\right)\begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix}\overline{\left(CA\overline{C}^t\right)}^t \\
=\ & CA\overline{C}^t\left(C\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\overline{C}^t\right)C\,\overline{A}^t\overline{C}^t \\
=\ & CA\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\overline{A}^t\overline{C}^t \\
=\ & C\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\overline{C}^t \\
=\ & \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix}.
\end{aligned}$$

This implies $N_n(1,1)$ is conjugate to

$$CN_n(1,1)\overline{C}^t = \mathrm{SU}(n,n).$$

2.3.4 The case of $m = 1, l = 2$

$N_n(1,2)$ is the group of all $\mathbb{H}$-linear transformations of $\mathbb{H}^{2n}$, which preserve the symplectic structure ω_1^L.

$$N_n(1,2) = \left\{ A + jB \in \mathrm{SL}(2n,\mathbb{H}) \mid (A+jB)\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \left(\overline{A+jB}\right)^t = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \right\}$$

Denote C the matrix $\frac{1}{2}\begin{pmatrix} (i+j)I & (-1-k)I \\ (-1+k)I & (i-j)I \end{pmatrix}$, we have

$$C\,\overline{C}^t = I$$

and

$$C\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\overline{C}^t = -j\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

So the group $N_n(1,2)$ is conjugate to

$$CN_n(1,2)\overline{C}^t = \left\{ A + jB \in \mathrm{SL}(2n,\mathbb{H}) \mid (A+jB)\; jI\; \left(\overline{A+jB}\right)^t = jI \right\}.$$

We use the embedding

$$\mathrm{SL}(2n,\mathbb{H}) \longrightarrow \mathrm{SL}(4n,\mathbb{C})$$
$$A + jB \mapsto \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix},$$

any element $A + jB \in CN_n(1,2)\overline{C}^t$ corresponds to $\begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \in \mathrm{SL}(4n,\mathbb{C})$, which satisfies

$$\begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\overline{\begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}}^t = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

By elementary calculation this is equivalent to

$$\begin{cases} A \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \overline{A}^t = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \\ AA^t = I, \end{cases}$$

where $A \in \mathrm{SL}(4n, \mathbb{C})$. That is $A \in \mathrm{SO}^*(4n)$. (See [9] p.445)

2.3.5 The case of $m = 2$

First we introduce the following three properties.

Let $(\mathfrak{g}, \sigma)$ be an orthogonal symmetric Lie algebra of noncompact semisimple type with standard decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}.$$

This implies that $\mathfrak{k}$ is the maximal compact Lie subalgebra of $\mathfrak{g}$. We define a real subspace $\mathfrak{g}^*$ of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ by

$$\mathfrak{g}^* = \mathfrak{k}^* + \mathfrak{m}^*, \quad \text{where } \mathfrak{k}^* = \mathfrak{k},\ \mathfrak{m}^* = \sqrt{-1}\mathfrak{m},$$

and σ^* a real linear map of $\mathfrak{g}^*$ into itself defined by

$$\sigma^*(X + Y) = X - Y, \qquad (X \in \mathfrak{k}^*, Y \in \mathfrak{m}^*).$$

Then $(\mathfrak{g}^*, \sigma^*)$ is an orthogonal symmetric Lie algebra of compact type, which is the compact dual to $(\mathfrak{g}, \sigma)$. $\mathfrak{g}^*$ is also a maximal compact Lie subalgebra of $\mathfrak{g}^{\mathbb{C}}$.

In the following we consider this on the Lie group level. If G is a Lie group of noncompact type with $\mathfrak{g}$ as its Lie algebra, and the maximal compact subgroup K with $\mathfrak{k}$ as its Lie algebra, then G/K is the noncompact dual to G_c/K, where G_c with $\mathfrak{g}_c$ as its Lie algebra is the maximal compact subgroup of the complexification of G.

Proposition 2.3.1 *$N_n(0,l)/G_n(0,l)$ is the noncompact dual to $G^s_n(1,l)/G_n(0,l)$.*

Proof. Consider $V^{[1,l]}$ together with the complex structures $J_1, ..., J_l$ and symplectic structure ω (in the proof of this and next propositions we omit R and L in J^R_j and ω^L_i respectively),

$$\omega(J_i u, J_i v) = \omega(u, v)$$

induces

$$\iota_\omega \circ J_i = J^*_i \circ \iota_\omega.$$

Let g be the standard Euclidean metric, by simple calculation we have

$$g(J_i u, J_i v) = g(u, v),$$

this induces

$$J_i \circ \iota_g^{-1} = \iota_g^{-1} \circ J^*_i.$$

We define a complex structure J (this is obvious a complex structure) by

$$\omega(u, v) = g(Ju, v),$$

i.e.

$$J = \iota_g^{-1} \circ \iota_\omega.$$

It is easy to check that

$$\begin{aligned} J_i \circ J &= J_i \circ \iota_g^{-1} \circ \iota_\omega \\ &= \iota_g^{-1} \circ J^*_i \circ \iota_\omega \\ &= \iota_g^{-1} \circ \iota_\omega \circ J_i \\ &= J \circ J_i, \end{aligned}$$

so

$$J_i \circ (J \circ A) = (J \circ A) \circ J_i.$$

This implies that the subgroup of the special linear group of $V^{[1,l]}$ whose elements preserve $J_1, ..., J_l$ and J is the complexification of the subgroup of the special linear group of $V^{[o,l]}$ whose elements preserve $J_1, ..., J_l$, i.e.

$$\mathrm{SL}(V^{[1,l]}, J_1, ..., J_l, J) = \mathrm{SL}(V^{[0,l]}, J_1, ..., J_l)^{\mathbb{C}}.$$

For any $A \in \mathrm{SL}(V^{[1,l]}, J_1, ..., J_l)$, if $g(Au, Av) = g(u, v)$, then

$$\begin{aligned} & A^* \circ \iota_\omega = \iota_\omega \circ A \\ \Leftrightarrow \quad & A \circ \iota_g^{-1} \circ \iota_\omega = \iota_g^{-1} \circ A^* \circ \iota_\omega = \iota_g^{-1} \circ \iota_\omega \circ A \\ \Leftrightarrow \quad & AJ = JA \end{aligned}$$

This implies

$$\mathrm{SL}(V^{[1,l]}, J_1, ..., J_l, J, g) = \mathrm{SL}(V^{[1,l]}, J_1, ..., J_l, \omega, g).$$

So

$$\begin{aligned} G_n(1, l) &= \mathrm{SL}(V^{[1,l]}, J_1, ..., J_l, \omega, g) \\ &= \mathrm{SL}(V^{[1,l]}, J_1, ..., J_l, J, g) \\ &= \text{maximal compact subgroup of } \mathrm{SL}(V^{[1,l]}, J_1, ..., J_l, J) \\ &= \text{maximal compact subgroup of } \mathrm{SL}(T^l V, J_1, ..., J_l)^{\mathbb{C}}, \end{aligned}$$

this implies our result. □

Proposition 2.3.2 *$N_n(2, l)$ is the complexification of $N_n(1, l)$.*

Proof. Let J' be $\iota_{\omega_1}^{-1} \circ \iota_{\omega_2}$, it is a complex structure since

$$\begin{aligned} (J')^2 &= (\iota_{\omega_1}^{-1} \circ \iota_{\omega_2})(\iota_{\omega_1}^{-1} \circ \iota_{\omega_2}) \\ &= (-\iota_{\omega_2}^{-1} \circ \iota_{\omega_1})(\iota_{\omega_1}^{-1} \circ \iota_{\omega_2}) \text{ (by preposition 2.2.1)} \\ &= -\iota_{\omega_2}^{-1} \circ \iota_{\omega_2} \\ &= -I. \end{aligned}$$

For any $A \in \mathrm{SL}(V^{[2,l]}, J_1, ..., J_l, \omega_1)$,

$$
\begin{aligned}
& \iota_{\omega_2} \circ A = A^* \circ \iota_{\omega_2} \\
\Leftrightarrow \quad & A \circ \iota_{\omega_1}^{-1} \circ \iota_{\omega_2} = \iota_{\omega_1}^{-1} \circ A^* \circ \iota_{\omega_2} = \iota_{\omega_1}^{-1} \circ \iota_{\omega_2} \circ A \\
\Leftrightarrow \quad & AJ' = J'A
\end{aligned}
$$

implies

$$\mathrm{SL}(V^{[2,l]}, J_1, ..., J_l, \omega_1, \omega_2) = \mathrm{SL}(V^{[2,l]}, J_1, ..., J_l, \omega_1, J').$$

Next we check $\mathrm{SL}(V^{[2,l]}, J_1, ..., J_l, \omega_1, J')$ is a complex Lie group, i.e. we need to check

$$J_i J' = J' J_i, \text{ for any } i = 1, ..., l$$

and

$$\omega_1(J'u, J'v) = -\omega_1(u, v), \text{ for any } u, v \in V^{[2,l]}.$$

First $J_i J' = J' J_i$ follows from

$$\iota_{\omega_1}^{-1} \circ \iota_{\omega_2} \circ J_i = \iota_{\omega_1}^{-1} \circ J_i^* \circ \iota_{\omega_2} = J_i \circ \iota_{\omega_1}^{-1} \circ \iota_{\omega_2}.$$

The following calculation

$$
\begin{aligned}
\omega_1(J'u, J'v) &= \omega_1(\iota_{\omega_1}^{-1} \circ \iota_{\omega_2}(u), \iota_{\omega_1}^{-1} \circ \iota_{\omega_2}(v)) \\
&= \iota_{\omega_1}(\iota_{\omega_1}^{-1} \circ \iota_{\omega_2}(u))(\iota_{\omega_1}^{-1} \circ \iota_{\omega_2}(v)) \\
&= \iota_{\omega_2}(u)(\iota_{\omega_1}^{-1} \circ \iota_{\omega_2}(v)) \\
&= \omega_2(u, \iota_{\omega_1}^{-1} \circ \iota_{\omega_2}(v)) \\
&= -\omega_2(\iota_{\omega_1}^{-1} \circ \iota_{\omega_2}(v), u) \\
&= -\iota_{\omega_2}(\iota_{\omega_1}^{-1} \circ \iota_{\omega_2}(v))(u) \\
&= -\iota_{\omega_2}(-\iota_{\omega_2}^{-1} \circ \iota_{\omega_1}(v))(u) \text{ (by proposition 2.2.1)} \\
&= \iota_{\omega_1}(v)(u) \\
&= -\omega_1(u, v)
\end{aligned}
$$

implies

$$\omega_1(J'u, J'v) = -\omega_1(u, v).$$

Unify the above discussion we have

$$\begin{aligned} N_n(l,2) &= \mathrm{SL}(V^{[2,l]}, J_1, ..., J_l, \omega_1, \omega_2) \\ &= \mathrm{SL}(V^{[2,l]}, J_1, ..., J_l, \omega_1, J') \\ &= \mathrm{SL}(V^{[1,l]}, J_1, ..., J_l, \omega_1)^{\mathbb{C}}, \end{aligned}$$

i.e. $\mathrm{N}_n(2,l)$ is the complexification of $N_n(1,l)$. □

Proposition 2.3.3 *$N_n(1,l)/G_n(1,l)$ is the noncompact dual of $G_n^s(2,l)/G_n(1,l)$.*

Proof. This property is a direct corollary of 2.3.2. □

Use proposition 2.3.2 we have the following table for $N_n(m,l)$

$\mathbb{A}\backslash\mathbb{B}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathbb{R}$	$\mathrm{SL}(n,\mathbb{R})$	$\mathrm{SL}(n,\mathbb{C})$	$\mathrm{SL}(n,\mathbb{H})$
$\mathbb{C}$	$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{SU}(n,n)$	$\mathrm{SO}^*(4n)$
$\mathbb{H}$	$\mathrm{Sp}(2n,\mathbb{C})$	$\mathrm{SL}(2n,\mathbb{C})$	$\mathrm{SO}(4n,\mathbb{C})$

The maximal compact (semisimple) subgroups $\mathrm{G}_n^s(m,l)$ of $N_n(m,l)$ are

$\mathbb{A}\backslash\mathbb{B}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathbb{R}$	$\mathrm{SO}(n)$	$\mathrm{SU}(n)$	$\mathrm{Sp}(n)$
$\mathbb{C}$	$\mathrm{SU}(n)$	$\mathrm{SU}(n)^2$	$\mathrm{SU}(2n)$
$\mathbb{H}$	$\mathrm{Sp}(n)$	$\mathrm{SU}(2n)$	$\mathrm{SO}(4n)$

At the Lie algebra level we also have the tables (T1) and (T2) in the introduction for the noncompact Lie algebras $\mathfrak{n}_n(m,l)$ and compact Lie algebras $L_n(\mathbb{A},\mathbb{B})$ respectively.

If we use the above properties 2.3.1, 2.3.2 and 2.3.3 to extend the above noncompact square to the fourth column, we have the table of noncompact square for $N_n(m,l)$ as in theorem 2.3.2.

Remark 2.3.1 *Now we will use the above properties to give another way to formulate the construction of the noncompact magic square of Lie groups.*

In the above construction we view $N_n(m,l)$ as the connected maximal semisimple subgroup of the group of all $\mathbb{B}$-linear transformations of $(T^*)^m\mathbb{B}^n$ which preserve ω_i^L (where $i = 1, ..., m$).

According to the proof of proposition 2.3.2, by introducing the complex structure $J_1^L = J' = \iota^{-1}_{\omega_1^L} \circ \iota_{\omega_2^L}$ (only when $m = 2$), $(T^*)^m\mathbb{B}^n$ can be identified as $\frac{\mathbb{A}}{2} \otimes \mathbb{B}$-linear subspace $(\frac{\mathbb{A}}{2} \otimes \mathbb{B})^{2n}$ when we only consider $\mathbb{R}$-linear transformations of $(T^*)^m\mathbb{B}^n$ which preserve the complex structures J_1^L (only when $m = 2$), $J_j^R = J_j$ (where $j = 1, ..., l$) and ω_m^L.

Now we view $N_n(m,l)$ as the connected maximal semisimple subgroup of the group of all $\frac{\mathbb{A}}{2} \otimes \mathbb{B}$-linear transformations of $(\frac{\mathbb{A}}{2} \otimes \mathbb{B})^{2n}$ which preserve ω_m^L, where ω_m^L is viewed as a $2n \times 2n$ matrix with entries in $\frac{\mathbb{A}}{2} \otimes \mathbb{B}$.

For any $A \in N_n(m,l)$, we identify $A \in SL(2n, \frac{\mathbb{A}}{2} \otimes \mathbb{B})$, which satisfies

$$A\omega_m^L \tilde{A}^t = \omega_m^L,$$

where $\tilde{}$ is the conjugation with respect to the right side of the tensor product, that is the conjugation of $\mathbb{B}$.

At the Lie algebra level, for any element $A \in \mathfrak{n}_n(m,l)$, identify $A \in M_{2n}(\frac{\mathbb{A}}{2} \otimes \mathbb{B})$, the Lie algebra of $2n \times 2n$ matrix with entries in $\frac{\mathbb{A}}{2} \otimes \mathbb{B}$, which satisfies

$$A\omega_m^L + \omega_m^L \tilde{A}^t = 0.$$

So by proper choosing of coordinates (such that ω_m^L is of the form $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ as a $2n \times 2n$ matrix with entries in $\frac{\mathbb{A}}{2} \otimes \mathbb{B}$), any element of $\mathfrak{n}_n(l,m)$ has the form

$$\begin{pmatrix} A & B \\ C & -\tilde{A}^t \end{pmatrix},$$

where $A, B, C \in M_n(\frac{\mathbb{A}}{2} \otimes \mathbb{B})$, the algebra of $n \times n$ matrices with entries in $\frac{\mathbb{A}}{2} \otimes \mathbb{B}$, and $\tilde{B}^t = B$, $\tilde{C}^t = C$.

Remark 2.3.2 *If we equip* $(\frac{\mathbb{A}}{2} \otimes \mathbb{B})^{2n}$ *with the standard Euclidean metric, the symplectic structure* ω_m^L *induces a complex structure* J_m^L*, the maximal compact subgroup* $G_n(m,l)$ *of* $N_n(m,l)$ *can be viewed as the group of all* $\mathbb{A} \otimes \mathbb{B}$*-linear transformations of* $(\mathbb{A} \otimes \mathbb{B})^n$ *in* $N_n(m,l)$ *which preserve the standard metric.*

From now on we consider $(\mathbb{A}\otimes\mathbb{B})^n$ instead of $(\frac{\mathbb{A}}{2}\otimes\mathbb{B})^{2n}$ unless stated otherwise, and use the above formulation of $G_n(m,l)$.

Chapter 3

Compact Riemannian Symmetric Spaces

3.1 Definitions and First Properties

Let (M, g) be a Riemannian manifold. Let $p \in M$ and let $B_r(p)$ be a normal coordinate ball around p. The diffeomorphism of $B_r(p)$

$$\sigma_p = \mathrm{Exp}_p \circ (-I_p) \circ \mathrm{Exp}_p^{-1},$$

is called the geodesic symmetry at p, where Exp_p is the exponential map at p and I_p is the identity map of T_pM, the tangent space of M at p.

Definition 3.1.1 *The Riemannian manifold (M, g) is said to be Riemannian locally symmetric if for each $p \in M$ there is a suitable $B_r(p)$ such that the geodesic symmetry σ_p is an isometry of $B_r(p)$ relative to the metric induced by g.*

Definition 3.1.2 *A Riemannian manifold (M, g) is a Riemannian symmetric space if each point $p \in M$ is an isolated fix point of an involutive isometry σ_p of M.*

Here "involutive" means that the square is the identity map and σ_p is called the symmetry at p. In this case, the differential $(\sigma_p)_{*p}$ is an linear isomorphism of T_pM which has no nonzero fixed vector, it must coincide with $-I_p$. Thus σ_p is the geodesic symmetry on every normal coordinate ball $B_r(p)$. Hence (M, g) is locally symmetric.

A topological space M is said to be simply connected if it is path connected and the fundamental group of M is trivial. A Riemannian manifold (M, g) is complete if it is complete as a metric space, i.e. every Cauchy sequence in M is a convergent sequence.

The following theorem gives a characterization of a Riemannian locally symmetric space in terms of the Riemannian curvature tensor field and a sufficient condition for a Riemannian locally symmetric space to be Riemannian symmetric.

Theorem 3.1.3 *Let (M, g) be a Riemannian manifold, D the Riemannian connection, and R the Riemannian curvature tensor field.*

1. *For (M, g) to be Riemannian locally symmetric it is necessary and sufficient for R to be parallel relative to $D : DR = 0$.*

2. *If (M, g) is a Riemannian locally symmetric space which is simply connected and complete, then it is Riemannian symmetric.*

The completeness assumption of the above theorem cannot be dropped, as we will see in the following theorem.

Theorem 3.1.4 *Let (M, g) be a Riemannian symmetric space. Then*

1. *(M, g) is complete.*

2. *(M, g) is Riemannian homogeneous.*

3. *The universal Riemannian covering manifold $(\tilde{M}, \tilde{g})$ of (M, g) is Riemannian symmetric.*

Two Riemannian manifolds (M_1, g_1) and (M_2, g_2) are said to be locally isometric if their universal Riemannian covering manifolds are isometric. In this thesis we will give a uniform description of Riemannian symmetric spaces up to local isometry. By theorem 3.1.4 this is equivalent to classifying simply connected Riemannian symmetric spaces up to isometry.

A simply connected Riemannian symmetric space (M, g) is uniquely decomposed as the direct product

$$(M, g) = (M_0, g_0) \times (M_1, g_1),$$

where (M_0, g_0) is a Euclidean space and (M_1, g_1) is a simply connected Riemannian symmetric space of semisimple type, which by definition has no parallel vector field, that is, the only smooth vector field on M_1 that is parallel relative to the Riemannian connection D is the zero vector field.

Because of the following de Rham decomposition, we can concentrate our attention on the irreducible Riemannian symmetric space.

Definition 3.1.5 *A Riemannian symmetric space (M, g) is said to be irreducible if, for each $p \in M$, the holonomy algebra $\mathfrak{h}(p)$ acts on T_pM irreducibly and nontrivially.*

Theorem 3.1.6 ***(de Rham decomposition)*** *A simply connected Riemannian symmetric space (M, g) is the direct product*

$$(M, g) = (M_0, g_0) \times (M_1, g_1) ... \times (M_m, g_m),$$

where (M_0, g_0) is a Euclidean space and each (M_i, g_i) is a simply connected irreducible Riemannian symmetric space. The decomposition is unique up to order of irreducible factors.

Let (M, g) be a connected Riemannian symmetric space, by theorem 3.1.4 (M, g) is Riemannian homogeneous. Let G be the identity component of the isometry group of (M, g), which acts transitively on M, K the isotropy subgroup of a fixed point $p \in M$. Hence M can be identified with the quotient space G/K together with the action of G by the correspondence

$$gK \mapsto g(p)$$

for $g \in G$.

The most well-known example of Riemannian symmetric space is the real Grassmannian $\mathrm{Gr}_{\mathbb{R}}(k, W)$, i.e. the space of all k-dimensional linear subspaces in a real vector space W. In this case, $\sigma_P(P')$ is the reflection of the k-dimensional subspace P' along P. Other Riemannian symmetric spaces can be constructed using (multi-)complex and (multi-) symplectic structures on W.

Recall when $\mathbb{A}, \mathbb{B} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and we write $\dim_{\mathbb{R}}\mathbb{A} = 2^m, \dim_{\mathbb{R}}\mathbb{B} = 2^l$, then $W = (\mathbb{A} \otimes \mathbb{B})^n$ can be identified as a real vector space together with complex structures J_i^L and J_j^R, where $i = 1, ..., m$ and $j = 1, ..., l$. If we equip W with the standard Euclidean metric g, then each complex structure J_i^L (respectively J_j^R) induces a symplectic structure ω_i^L (respectively ω_j^R) on W. Let $\mathrm{Gr}_{\mathbb{R}}(W)$ denote the space of all $\mathbb{R}$-linear subspaces of W, i.e.,

$$\begin{aligned} \mathrm{Gr}_{\mathbb{R}}(W) &= \coprod_{k=0}^{N} \mathrm{Gr}_{\mathbb{R}}(k, W) \\ &= \coprod_{k=0}^{N} \mathrm{O}(N)/\mathrm{O}(k)\mathrm{O}(N-k), \end{aligned}$$

where $N = \dim_{\mathbb{R}} W$.

Each complex structure J on W induces an involution σ_J on $\mathrm{Gr}_{\mathbb{R}}(W)$, given by

$$\sigma_J : \mathrm{Gr}_{\mathbb{R}}(W) \to \mathrm{Gr}_{\mathbb{R}}(W),$$

$$\sigma_J(P) = J(P) = \{J(v) | v \in P\},$$

for any $\mathbb{R}$-linear subspace $P \subset W$. Its fix point set

$$\mathrm{Gr}_{\mathbb{R}}(W)^{\sigma_J} = \{P \in \mathrm{Gr}_{\mathbb{R}}(W) \mid J(P) = P\}$$

is simply the union of complex Grassmannians of J-complex linear subspaces in W, i.e.

$$\mathrm{Gr}_{\mathbb{R}}(W)^{\sigma_J} \cong \coprod_{k=1}^{N} \frac{\mathrm{U}(N)}{\mathrm{U}(k)\mathrm{U}(N-k)},$$

where $2N = \dim_{\mathbb{R}} W$.

Similarly, any symplectic structure ω on W induces a map σ_ω on $\mathrm{Gr}_{\mathbb{R}}(W)$,

$$\sigma_\omega : \mathrm{Gr}_{\mathbb{R}}(W) \to \mathrm{Gr}_{\mathbb{R}}(W),$$

$$\sigma_\omega(P) = P^{\perp_\omega} = \{v \in W | \omega(v,w) = 0, \text{for any } w \in P\},$$

for any $\mathbb{R}$-linear subspace $P \subset W$. Non-degeneracy of ω implies that

$$\dim_{\mathbb{R}} P + \dim_{\mathbb{R}} \sigma_\omega(P) = \dim_{\mathbb{R}} W.$$

Together with the simple fact $P \subset (P^{\perp_\omega})^{\perp_\omega}$, they imply that σ_ω is also an involution, i.e. $(\sigma_\omega)^2 = id$. Observe that $\sigma_\omega(P) = P$ if and only if P is a Lagrangian subspace of W. Namely P is a half dimensional subspace such that ω vanishes on P. Hence the fix point set of σ_ω can be identified as the Lagrangian Grassmannian, i.e.

$$\mathrm{Gr}_{\mathbb{R}}(W)^{\sigma_\omega} \cong \frac{\mathrm{U}(N)}{\mathrm{O}(N)}.$$

Proposition 3.1.1 *Given any Hermitian vector space (W, g, J, ω) of dimension $2n$, the involutions σ_J and σ_ω are isometries on $Gr_{\mathbb{R}}(2k, W)$ $(k = 1, ..., n)$ and $Gr_{\mathbb{R}}(n, W)$ respctively.*

Proof. Take an orthonormal real basis $\{e_1, e_2, ..., e_{2n}\}$ of W, such that

$$Je_{2i-1} = e_{2i}, Je_{2i} = -e_{2i-1},$$

for $i = 1, ..., n$. For any $A \in \mathrm{O}(2n)$, define

$$\begin{aligned} \sigma_J : \mathrm{Gr}_{\mathbb{R}}(2k, W) &\to \mathrm{Gr}_{\mathbb{R}}(2k, W), \\ \sigma_J(AK) &= AJK, \end{aligned}$$

where K is $\mathrm{O}(2k)\mathrm{O}(2n-2k)$. σ_J is well-defined because the linear subspaces generated by $\{e_1, e_2, ..., e_{2k}\}$ and $\{e_{2k+1}, e_{2k+2}, ..., e_{2n}\}$ are complex. Since $J \in \mathrm{O}(W)$, and $\mathrm{O}(W)$ is the isometry group of $\mathrm{Gr}_{\mathbb{R}}(2k, W)$, this implies σ_J is an isometry with respect to the invariant metric of $\mathrm{Gr}_{\mathbb{R}}(2k, W)$.

For σ_ω, we choose an orthonormal basis such that ω (as a non-degenerate two form) is of the form

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

For any $A \in \mathrm{O}(2n)$, define

$$\begin{aligned} \sigma_\omega : \mathrm{Gr}_{\mathbb{R}}(n, W) &\rightarrow \mathrm{Gr}_{\mathbb{R}}(n, W), \\ \sigma_\omega(AK) &= \omega A \omega^{-1} K, \end{aligned}$$

where $K = \mathrm{O}(n)\mathrm{O}(n)$. It is easy to check σ_ω is also well-defined.

$$\begin{aligned} & \omega(A(e_1, ..., e_{2n})^t, \omega A \omega^{-1}(e_1, ..., e_{2n})^t) \\ = \; & A\omega(\omega A \omega^{-1})^t \\ = \; & \omega \\ = \; & \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \end{aligned}$$

Here we view the upper n rows of A as a basis of the linear subspace corresponding to A, this implies that for any linear subspace P corresponding to A, $\sigma_\omega(P) = P^{\perp_\omega}$ is represented by $\omega A \omega^{-1}$, i.e. the upper n rows of $\omega A \omega^{-1}$ is a basis of the linear subspace $P^{\perp_\omega}$. As a matrix ω is in $\mathrm{O}(2n)$, this implies σ_ω is an isometry on $\mathrm{Gr}_{\mathbb{R}}(n, W)$. □

The following lemma enables us to construct many symmetric spaces from the real Grassmannian using involutive isometries induced by complex and symplectic structures on W.

Lemma 3.1.1 *Let M be a Riemannian symmetric space and $\Sigma = \{\sigma_1, \sigma_2, ..., \sigma_s\}$ be a collection of involutive isometries, then the fix point set*

$$M^\Sigma = \bigcap_{\sigma\in\Sigma} M^\sigma$$

with the induced Riemannian metric is again a Riemannian symmetric space.

Proof. Without loss of generality we may assume that M^Σ is connected. First it is easy to see that M^σ is a closed submanifold of M for any σ in Σ. If we can show that the second fundamental form B of the Levi-Civita connection D is zero, then M^σ is a totally geodesic submanifold, so M^Σ (intersection of totally geodesic submanifolds) is also a totally geodesic submanifold. Every geodesic in M which passes through a point x in M^Σ with the tangent in T_xM^Σ is still a geodesic in M^Σ, this implies that M^Σ is a locally symmetric space with the induced metric. For any geodesic isometry σ_x of M, where $x \in M^\Sigma$, $\sigma_x(M^\Sigma) \subset M^\Sigma$ (because σ_x is just reverse the geodesic) and M^Σ is closed in M, so M^Σ is a global symmetric space.

Now we prove the second fundamental form B is zero. For any point $x \in M^\sigma$, we have the decomposition of vector space $T_xM = T_xM^\sigma \oplus N_x$, where N_x is the normal vector space of M^σ at x. σ is an isometry, so

$$D_X(Y) = \sigma_*(D_X(Y)) = D_{\sigma_*X}(\sigma_*Y),$$

where X, Y are vectors of M at x. Denote D', D'' as the connections induced by D on TM^σ and the normal bundle N of M^σ in M, and e, f as the basis of TM^σ and N respectively, we have

$$\begin{pmatrix} D'_e(e) & B_e(f) \\ -B^t_f(e) & D''_f(f) \end{pmatrix} = \begin{pmatrix} D'_e(e) & -B_e(f) \\ -(-B^t_f(e)) & D''_f(f) \end{pmatrix}.$$

This obviously implies $B = 0$, i.e. the second fundamental form is zero. □

Back to $W = (\mathbb{A} \otimes \mathbb{B})^n$, we denote the set of its canonical complex structures induced by $\mathbb{A}$ as $\Sigma^L = \left\{J^L_1, ..., J^L_{l-1}, J^L_l\right\}$. We also define $\hat{\Sigma}^L = \left\{J^L_1, ..., J^L_{l-1}, \omega^L_l\right\}$

and we have Σ^R and $\hat{\Sigma}^R$ defined in a similar fashsion using complex and symplectic structures induced from $\mathbb{B}$. They define involutive isometries σ's on $\mathrm{Gr}_{\mathbb{R}}(W)$ whose fixed points correspond to complex subspaces or Lagrangian subspaces in W.

Definition 3.1.7 *Let $W = (\mathbb{A} \otimes \mathbb{B})^n$, where $\mathbb{A}, \mathbb{B} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.*

1. *Grassmannians $Gr_{\mathbb{A}\mathbb{B}}(k, n)$: The connected component containing $(\mathbb{A} \otimes \mathbb{B})^k$ of*

$$\bigcap_{I \in \Sigma^L \cup \Sigma^R} Gr_{\mathbb{R}}(k \cdot 2^{m+l}, W)^{\sigma_I}.$$

2. *Lagrangian Grassmannians $LGr_{\mathbb{A}\mathbb{B}}(n)$: The connected component containing $(\frac{\mathbb{A}}{2} \otimes \mathbb{B})^n$ of (with the Abelian part taken away)*

$$\bigcap_{I \in \hat{\Sigma}^L \cup \Sigma^R} Gr_{\mathbb{R}}(n \cdot 2^{m+l-1}, W)^{\sigma_I}.$$

3. *Double Lagrangian Grassmannians $LLGr_{\mathbb{A}\mathbb{B}}(n)$: The connected component containing $(\frac{\mathbb{A}}{2} \otimes \frac{\mathbb{B}}{2})^n \oplus (j_1\frac{\mathbb{A}}{2} \otimes j_2\frac{\mathbb{B}}{2})^n$ of (with the Abelian part taken away)*

$$\bigcap_{I \in \hat{\Sigma}^L \cup \hat{\Sigma}^R} Gr_{\mathbb{R}}(n \cdot 2^{m+l-1}, W)^{\sigma_I}.$$

By lemma 3.1.1, these Grassmannians, Lagrangian Grassmannians and Double Lagrangian Grassmannians are Riemannian symmetric spaces.

3.2 Grassmannians

$\mathrm{Gr}_{\mathbb{A}\mathbb{B}}(k,n)$ can be viewed as the connected component containing $(\mathbb{A}\otimes\mathbb{B})^k$ of the space of all $\mathbb{A}\otimes\mathbb{B}$-linear subspaces P in $W=(\mathbb{A}\otimes\mathbb{B})^n$ with $\dim_{\mathbb{A}\otimes\mathbb{B}}P=k$, where $\mathbb{A},\mathbb{B}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$.

Proposition 3.2.1 *When* $\mathbb{A},\mathbb{B}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$, $Gr_{\mathbb{A}\mathbb{B}}(k,n)$ *are given by the spaces in the table (C2) in the introduction.*

Proof. Denote the invariant subspace $(\mathbb{A}\otimes\mathbb{B})^k$ by P_0. It is easy to see that for any $A\in G_n(m,l)$, $A\cdot P_0$ is also an invariant linear subspace under all of the involutions, because A preserves all these complex structures. The $G_n(m,l)$ orbit of P_0 is connected. Note $\mathrm{O}(n)/\mathrm{O}(k)\mathrm{O}(n-k)$ is connected though $\mathrm{O}(n)$ is not connected. So the $G_n(m,l)$ orbit of P_0 is contained in $\mathrm{Gr}_{\mathbb{A}\mathbb{B}}(k,n)$. Therefore, it is enough to show that both have the same dimension.

Let

$$\begin{pmatrix} I & B \end{pmatrix}$$

be the coordinates of a point in $\mathrm{Gr}_{\mathbb{A}\mathbb{B}}(k,n)$ (k row vectors in $(\mathbb{A}\otimes\mathbb{B})^n$ as a basis of an invariant subspace), so B is a $k\times(n-k)$ matrix with entries in $\mathbb{A}\otimes\mathbb{B}$. This implies that the dimension of $\mathrm{Gr}_{\mathbb{A}\mathbb{B}}(k,n)$ is the same as the dimension of the $G_n(m,l)$ orbit. It is easy to see that the isotropy subgroup of P_0 is $G_k(m,l)G_{n-k}(m,l)$.

□

Corollary 3.2.1 *When* $\mathbb{A},\mathbb{B}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$, *we have*

$$Gr_{\mathbb{A}\mathbb{B}}(k,n)=G/K,$$

where

$$Lie\,(G)=\mathfrak{g}_n(m,l),\ and$$
$$Lie\,(K)=\mathfrak{g}_k(m,l)\oplus\mathfrak{g}_{n-k}(m,l).$$

When $\mathbb{A}$ or/and $\mathbb{B}$ equals $\mathbb{O}$, $\mathrm{Gr}_{\mathbb{AB}}(k,n)$ need to be defined with more care because of the non-existence of $\mathbb{O}^n$. Nevertheless, with the help of Jordan algebras, we can make sense of the Lie algebra $L_n(\mathbb{O})$ of infinitesimal symmetries of it when $n=2$ or 3. Namely they are Spin(9) and F_4 respectively. Similarly, using the magic square, we can define the corresponding Lie algebra $L_n(\mathbb{A},\mathbb{B})$ for any normed division algebras $\mathbb{A}$ and $\mathbb{B}$, provided that $n=2,3$. It is therefore natural to extend the definition of $\mathrm{Gr}_{\mathbb{AB}}(k,n)$ to include the octonion numbers as below:

Theorem 3.2.1 *(Definition-Theorem): Suppose $\mathbb{A}$ and $\mathbb{B}$ are any normed division algebras.*

We define $Gr_{\mathbb{AB}}(k,n)$ to be the symmetric space G/K with

$$Lie\,(G) = L_n(\mathbb{A},\mathbb{B}) = Der(\mathbb{A}) \oplus Der(\mathbb{B}) \oplus A'_n(\mathbb{A}\otimes\mathbb{B}), \text{ and}$$

$$Lie\,(K) = Der(\mathbb{A}) \oplus Der(\mathbb{B}) \oplus A'_k(\mathbb{A}\otimes\mathbb{B}) \oplus A'_{n-k}(\mathbb{A}\otimes\mathbb{B}) \oplus Im(\mathbb{A}) \oplus Im(\mathbb{B}),$$

such that if $\mathbb{A},\mathbb{B} \in \{\mathbb{R},\mathbb{C},\mathbb{H}\}$, then $Gr_{\mathbb{AB}}(k,n)$ is

$$\bigcap_{I\in\Sigma^L\cup\Sigma^R} Gr_{\mathbb{R}}\left(\overline{k}, (\mathbb{A}\otimes\mathbb{B})^n\right)^{\sigma_I}$$

containing $(\mathbb{A}\otimes\mathbb{B})^k$. Here $\overline{k} = dim_{\mathbb{R}}(\mathbb{A}\otimes\mathbb{B})^k = k\cdot 2^{m+l}$.

If $\mathbb{A}$ or $\mathbb{B}$ equals $\mathbb{O}$, then we assume that n equals 3.

Remark 3.2.1 *Here we use the Vinberg's version of the magic square [16], and similar in the subsequence.*

Proof. This follows immediately from proposition 3.2.1 and its corollary 3.2.1. □

Remark 3.2.2 *These types of exceptional symmetric spaces are described as projective planes over $\mathbb{A}\otimes\mathbb{B}$ (see [3]). In fact we can construct the projective planes $\mathbb{OP}^2$ and $(\mathbb{C}\otimes\mathbb{O})\mathbb{P}^2$ by using Jordan algebras over $\mathbb{R}$ and $\mathbb{C}$ respectively, but there is no geometric construction for $(\mathbb{H}\otimes\mathbb{O})\mathbb{P}^2$ and $(\mathbb{O}\otimes\mathbb{O})\mathbb{P}^2$ because $\mathbb{H},\mathbb{O}$ are not fields, thus these is no Jordan algebra defined over them.*

Theorem 3.2.2 *For any normed division algebras* $\mathbb{A}, \mathbb{B} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, $Gr_{\mathbb{A}\mathbb{B}}(k, n)$ *are given by the spaces in the table (C2) in the introduction.*

3.3 Lagrangian Grassmannians

$\mathrm{LGr}_{\mathbb{AB}}(n)$ can be viewed as the connected component containing $\left(\frac{\mathbb{A}}{2}\otimes\mathbb{B}\right)^n$ of the space of all $\frac{\mathbb{A}}{2}\otimes\mathbb{B}$-linear subspaces P in $W=(\mathbb{A}\otimes\mathbb{B})^n$ with $\dim_{\frac{\mathbb{A}}{2}\otimes\mathbb{B}}P=n$ invariant under $\sigma_{\omega_m^L}$, where $\mathbb{A}\in\{\mathbb{C},\mathbb{H}\}$ and $\mathbb{B}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$.

Proposition 3.3.1 *When $\mathbb{A}\in\{\mathbb{C},\mathbb{H}\}$ and $\mathbb{B}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$, $LGr_{\mathbb{AB}}(n)$ are given by the spaces in the table (C3) in the introduction.*

Proof. Recall $\mathrm{LGr}_{\mathbb{AB}}(n)$ is the space of all $\mathbb{R}$-linear subspaces of W invariant under the involutions $\sigma_{J_i^L},\sigma_{J_j^R}$ and $\sigma_{\omega_m^L}$, where $i=1,...,m-1$, $j=1,..,l$. Denote the invariant subspace $\left(\frac{\mathbb{A}}{2}\otimes\mathbb{B}\right)^n$ by P_0. As in the Grassmannians case, we prove that the $G_n^s(m,l)$ orbit of P_0 is the connected component containing P_0. The orbit of P_0 is obviously contained in the connected component, thus it is enough to show that both have same dimension.

Let

$$\left(\begin{array}{c} I, jB \end{array}\right)$$

be the coordinates of a point in $\mathrm{LGr}_{\mathbb{AB}}(n)$ (n row vectors in $W=\left(\frac{\mathbb{A}}{2}\otimes\mathbb{B}\right)^n\oplus\left(j\cdot\frac{\mathbb{A}}{2}\otimes\mathbb{B}\right)^n$ as a basis of an invariant subspace P), where jB is a $n\times n$ matrix with entries in $j\cdot\frac{\mathbb{A}}{2}\otimes\mathbb{B}$. We have

$$(I,jB)\;\omega_m^L\;\left(I,j\tilde{B}\right)^t=0,$$

where $B\in M_n\left(\frac{\mathbb{A}}{2}\otimes\mathbb{B}\right)$, it implies:

$$\tilde{B}^t=B.$$

So the dimension of this component is the same as the $G_n^s(m,l)$ orbit (see remark 2.3.1). □

Corollary 3.3.1 *When $\mathbb{A}\in\{\mathbb{C},\mathbb{H}\}$ and $\mathbb{B}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$, we have*

$$LGr_{\mathbb{AB}}(n)=G/K,$$

where

$$\begin{aligned}
Lie\,(G) &= L_n(\mathbb{A},\mathbb{B}),\, and \\
Lie\,(K) &= L_n\left(\frac{\mathbb{A}}{2},\mathbb{B}\right) \oplus Im\left(\frac{\mathbb{A}}{2}\right).
\end{aligned}$$

When $\mathbb{A}$ or/and $\mathbb{B}$ equals $\mathbb{O}$, the definition for $\mathrm{LGr}_{\mathbb{AB}}(n)$ is naturally extended as below.

Theorem 3.3.1 *(Definition-Theorem): Suppose* $\mathbb{A}$ *and* $\mathbb{B}$ *are any normed division algebras and* $\mathbb{A}$ *is not* $\mathbb{R}$*.*

We define $LGr_{\mathbb{AB}}(n)$ *to be the symmetric space* G/K *with*

$$\begin{aligned}
Lie\,(G) &= L_n(\mathbb{A},\mathbb{B}) = Der(\mathbb{A}) \oplus Der(\mathbb{B}) \oplus A'_n(\mathbb{A}\otimes\mathbb{B}),\, and \\
Lie\,(K) &= L_n\left(\frac{\mathbb{A}}{2},\mathbb{B}\right) \oplus Im\left(\frac{\mathbb{A}}{2}\right),
\end{aligned}$$

such that if $\mathbb{A},\mathbb{B} \in \{\mathbb{R},\mathbb{C},\mathbb{H}\}$*, then* $LGr_{\mathbb{AB}}(n)$ *is the connected component containing* $\left(\frac{\mathbb{A}}{2}\otimes\mathbb{B}\right)^n$ *of (with the Abelian part taken away)*

$$\bigcap_{I\in\hat{\Sigma}^L\cup\Sigma^R} Gr_{\mathbb{R}}\left(\overline{k},(\mathbb{A}\otimes\mathbb{B})^n\right)^{\sigma_I},$$

where $\overline{k} = \frac{dim_{\mathbb{R}}(\mathbb{A}\otimes\mathbb{B})^n}{2} = n\cdot 2^{m+l-1}$.

If $\mathbb{A}$ *or* $\mathbb{B}$ *equals* $\mathbb{O}$*, then we assume that* n *equals* 3*.*

Proof. This follows immediately from proposition 3.3.1 and its corollary 3.3.1. □

Theorem 3.3.2 *For any normed division algebra* $\mathbb{A} \in \{\mathbb{C},\mathbb{H},\mathbb{O}\}$ *and* $\mathbb{B} \in \{\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}\}$*,* $LGr_{\mathbb{AB}}(n)$ *are given by the spaces in the table (C3) in the introduction.*

3.4 Double Lagrangian Grassmannians

$\mathrm{LLGr}_{\mathbb{AB}}(n)$ can be viewed as the connected component containing $\left(\frac{\mathbb{A}}{2} \otimes \frac{\mathbb{B}}{2}\right)^n \oplus \left(j_1\frac{\mathbb{A}}{2} \otimes j_2\frac{\mathbb{B}}{2}\right)^n$ of the space of all $\frac{\mathbb{A}}{2}\otimes\frac{\mathbb{B}}{2}$-linear subspaces P in $W = (\mathbb{A}\otimes\mathbb{B})^n$ invariant under $\sigma_{\omega_m^L}$ and $\sigma_{\omega_l^R}$ with $\dim_{\frac{\mathbb{A}}{2}\otimes\frac{\mathbb{B}}{2}} P = 2n$, where $\mathbb{A} \in \{\mathbb{C}, \mathbb{H}\}$ and $\mathbb{B} \in \{\mathbb{C}, \mathbb{H}\}$.

Proposition 3.4.1 *When $\mathbb{A}, \mathbb{B} \in \{\mathbb{C}, \mathbb{H}\}$, $LLGr_{\mathbb{AB}}(n)$ are given by the spaces in the table (C4) in the introduction.*

Proof. The $G_n^s(m, l)$ orbit of $P_0 = \left(\frac{\mathbb{A}}{2} \otimes \frac{\mathbb{B}}{2}\right)^n \oplus \left(j_1\frac{\mathbb{A}}{2} \otimes j_2\frac{\mathbb{B}}{2}\right)^n$ is contained in $\mathrm{LLGr}_{\mathbb{AB}}(n)$ since $A \cdot P_0$ is also an invariant subspace in W for any $A \in G_n^s(m, l)$. Thus we only need to prove both have same dimensions as before.

First we determine the Lie algebra $\mathfrak{k}$ of the isotropy subgroup of P_0 in $\mathrm{Gr}_{\mathbb{R}}\left(\overline{k}, W\right)$, it is easy to see that $\mathfrak{k}$ is

$$A_n\left(\frac{\mathbb{A}}{2} \otimes \frac{\mathbb{B}}{2}\right) \oplus A_n\left(j_1\frac{\mathbb{A}}{2} \otimes j_2\frac{\mathbb{B}}{2}\right).$$

1. When $m = l = 1$, $\mathfrak{k}$ is obvious $\mathfrak{so}(n)^2$.

2. When $m = 1, l = 2$. Any element of $A_n(\mathbb{C} \otimes \mathbb{R}) \oplus A_n(j\mathbb{C} \otimes i\mathbb{R})$ can be represented as

   $$A \cdot 1 \otimes 1 + B \cdot i \otimes 1 + C \cdot j \otimes i + D \cdot k \otimes i,$$

   where $A, B, C, D \in M_n(\mathbb{R})$. These can be identify as

   $$\begin{pmatrix} A - D & C - B \\ C + B & A + D \end{pmatrix}.$$

   It is not difficult to check that this is an element in $\mathfrak{so}(2n, \mathbb{R})$, so $A_n(\mathbb{C} \otimes \mathbb{R}) \oplus A_n(j\mathbb{C} \otimes i\mathbb{R})$ is isomorphic to $\mathfrak{so}(2n)$.

3. When $m = 2, l = 1$, this is the same case as $m = 1, l = 2$.

4. When $m = l = 2$, $\mathfrak{k}$ is $A_n(\mathbb{C}\otimes\mathbb{C}) \oplus A_n(j\mathbb{C}\otimes j\mathbb{C})$. $\mathbb{C}\otimes\mathbb{C}$ can be identified as

$$\mathbb{C}\oplus\mathbb{C},$$

this implies that $\mathfrak{k}$ is isomorphic to

$$(A_n(\mathbb{C}) \oplus A_n(j\otimes j\cdot\mathbb{C})) \oplus (A_n(\mathbb{C}) \oplus A_n(j\otimes j\cdot\mathbb{C})).$$

Then any element of $\mathfrak{k}$ can be represented as

$$A + B\cdot i + C\cdot j\otimes j + D\cdot k\otimes j,$$

where $A, B, C, D \in M_n(\mathbb{R})$, then identify this to

$$\begin{pmatrix} A-D & C-B \\ C+B & A+D \end{pmatrix},$$

so k is isomorphic to $\mathfrak{so}(2n)^2$.

To show that the orbit is the connected component we only need to consider the dimensions of the orbit and the component because the orbit is contained in the connected component.

We use $\{x, y, z, w\}$ to denote the coordinates of W with respect to the following decomposition:

$$\left(\frac{\mathbb{A}}{2}\otimes\frac{\mathbb{B}}{2}\right)^n \oplus \left(j_1\frac{\mathbb{A}}{2}\otimes j_2\frac{\mathbb{B}}{2}\right)^n \oplus \left(j_1\frac{\mathbb{A}}{2}\otimes\frac{\mathbb{B}}{2}\right)^n \oplus \left(\frac{\mathbb{A}}{2}\otimes j_2\frac{\mathbb{B}}{2}\right)^n.$$

We use

$$\begin{pmatrix} I & 0 & j_1B_1 & 0 \\ 0 & I & 0 & j_2B_2 \end{pmatrix}$$

to denote any point P near P_0, where $B_1, B_2 \in M_n\left(\frac{\mathbb{A}}{2}\otimes\frac{\mathbb{B}}{2}\right)$. Since P is an invariant subspace, we have the following conditions:

$$\begin{pmatrix} I & 0 & j_1B_1 & 0 \\ 0 & I & 0 & j_2B_2 \end{pmatrix} \omega_m^L \begin{pmatrix} I & 0 & j_1\tilde{B}_1 & 0 \\ 0 & I & 0 & j_2\tilde{B}_2 \end{pmatrix}^t = 0,$$

$$\begin{pmatrix} I & 0 & j_1B_1 & 0 \\ 0 & I & 0 & j_2B_2 \end{pmatrix} \omega_l^R \begin{pmatrix} I & 0 & j_1\overline{\tilde{B}_1} & 0 \\ 0 & I & 0 & j_2\overline{\tilde{B}_2} \end{pmatrix}^t = 0.$$

Under the above coordinates, ω_l^R, ω_m^L is represented by the following matrices:

$$\omega_l^R = \begin{pmatrix} 0 & 0 & 0 & I \\ 0 & 0 & -I & 0 \\ 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \end{pmatrix}, \quad \omega_m^L = \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & -I \\ -I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix}.$$

Denote

$$P_1 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

so we have

$$B \cdot P_1 + P_1 \cdot \ B^t = 0, \ P_2 \cdot \ B^t = B \cdot P_2.$$

By simple calculation we have

$$B = \begin{pmatrix} B_1 & B_2 \\ -B_2 & -B_1^t \end{pmatrix},$$

where $B_1^t = B_1, B_2^t = B_2$.

When $m = l = 1$, the dimension is $n^2 - n$.

When $m = 1, l = 2$ or $m = 1, l = 2$, we have

$$J_i^R B_1 = B_1 J_i^R, J_j^L B_1 = B_2 J_j^L,$$

so

$$B_1 = \begin{pmatrix} a & b \\ b & -a \end{pmatrix},$$

where $a^t = a, b^t = b$, so the dimension of B_1 is $n^2 + n$. By the same way, the dimension of B_2 is n^2, so the result is true.

For $m = l = 2$, the case is similar. □

Corollary 3.4.1 *When* $\mathbb{A}, \mathbb{B} \in \{\mathbb{C}, \mathbb{H}\}$*, we have*

$$LLGr_{\mathbb{A}\mathbb{B}}(n) = G/K,$$

where

$$Lie\,(G) = L_n(\mathbb{A},\mathbb{B}), and$$
$$Lie\,(K) = A_n\left(\frac{\mathbb{A}}{2}\otimes\frac{\mathbb{B}}{2}\right)\oplus A_n\left(j_1\frac{\mathbb{A}}{2}\otimes j_2\frac{\mathbb{B}}{2}\right).$$

When $\mathbb{A}$ or/and $\mathbb{B}$ equals $\mathbb{O}$, the definition for $\mathrm{LLGr}_{\mathbb{AB}}(n)$ is naturally extended as below.

Theorem 3.4.1 *(Definition-Theorem): Suppose* $\mathbb{A}$ *and* $\mathbb{B}$ *are any normed division algebras except* $\mathbb{R}$.

We define $LLGr_{\mathbb{AB}}(n)$ *to be the symmetric space* G/K *with*

$$Lie\,(G) = L_n(\mathbb{A},\mathbb{B}) = Der(\mathbb{A})\oplus Der(\mathbb{B})\oplus A_n'(\mathbb{A}\otimes\mathbb{B}), and$$
$$Lie\,(K) = Der\left(\frac{\mathbb{A}}{2}\right)\oplus Der\left(\frac{\mathbb{B}}{2}\right)\oplus A_n\left(\frac{\mathbb{A}}{2}\otimes\frac{\mathbb{B}}{2}\right)\oplus A_n\left(j_1\frac{\mathbb{A}}{2}\otimes j_2\frac{\mathbb{B}}{2}\right),$$

such that if $\mathbb{A},\mathbb{B}\in\{\mathbb{C},\mathbb{H}\}$, *then* $LLGr_{\mathbb{AB}}(n)$ *is the connected component containing* $\left(\frac{\mathbb{A}}{2}\otimes\frac{\mathbb{B}}{2}\right)^n\oplus\left(j_1\frac{\mathbb{A}}{2}\otimes j_2\frac{\mathbb{B}}{2}\right)^n$ *of (with the Abelian part taken away)*

$$\bigcap_{I\in\hat{\Sigma}^L\cup\hat{\Sigma}^R} Gr_{\mathbb{R}}\left(\overline{k},(\mathbb{A}\otimes\mathbb{B})^n\right)^{\sigma_I},$$

where $\overline{k} = \frac{dim_{\mathbb{R}}(\mathbb{A}\otimes\mathbb{B})^n}{2} = n\cdot 2^{m+l-1}$.

If $\mathbb{A}$ *or* $\mathbb{B}$ *equals* $\mathbb{O}$, *then we assume that* n *equals* 3.

Proof. This follows immediately from proposition 3.4.1 and its corollary 3.4.1. □

Theorem 3.4.2 *For any normed division algebras* $\mathbb{A},\mathbb{B}\in\{\mathbb{C},\mathbb{H},\mathbb{O}\}$, $LLGr_{\mathbb{AB}}(n)$ *are given by the spaces in the table (C4) in the introduction.*

Remark 3.4.1 *When* $n=3$, *the double Lagrangian Grassmannians can be viewed as*

$$LLGr_{\mathbb{AB}} = \left\{\bigwedge^2\left(\frac{\mathbb{A}}{2}\otimes\frac{\mathbb{B}}{2}\right)^4\subset(\mathbb{A}\otimes\mathbb{B})^3\right\}.$$

3.5 Compact Simple Lie Groups

Let $W = (\mathbb{A} \otimes \mathbb{B})^n$ with complex structures J_i^L, J_j^R as before, $T^*W = W + W^*$ is a vector space with the induced complex structures J_i^L, J_j^R and a symplectic structure $\omega = dx \wedge dy$ (here $\{x, y\}$ denote the coordinates of W and the cotangent space respectively). But here we consider the symmetric tensor (non-definite)

$$g' = dx \otimes dy$$

rather than the skew-symmetric structure ω.

As in the case of ω, g' induces an involution on $\mathrm{Gr}_{\mathbb{R}}(T^*W)$,

$$\sigma_{g'} : \mathrm{Gr}_{\mathbb{R}}(T^*W) \to \mathrm{Gr}_{\mathbb{R}}(T^*W),$$

$$\sigma_{g'}(P) = P^{\perp_{g'}} = \{v \in T^*W | g'(v, w) = 0, \text{for any } w \in P\},$$

for any real linear subspace $P \subset T^*W$.

Let N be the dimension of W over $\mathbb{R}$, for any $A \in \mathrm{O}(2N)$, $\sigma_{g'}$ is defined as

$$\sigma_{g'}(AK) \;=\; g'Ag'^{-1}K,$$

where K is $\mathrm{O}(N)\mathrm{O}(N)$ and g' is the matrix $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, where I is the identity matrix in $\mathrm{O}(N)$. So σ_ω is also well-defined and is an isometry on $\mathrm{Gr}_{\mathbb{R}}(N, 2N)$.

Denote the connected component containing W (with the Abelian part taken away) of the space of all linear subspaces of T^*W which are invariant under the involutive isometries $\sigma_{J_i^L}, \sigma_{J_j^R}$ and $\sigma_{g'}$ as $G_{\mathbb{AB}}(n)$, where $i = 1, ..., m, j = 1, ..., l$.

Proposition 3.5.1 *When $\mathbb{A}, \mathbb{B} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, $G_{\mathbb{AB}}(n)$ is the space in the table (C1) in the introduction.*

Remark 3.5.1 *These symmetric spaces are compact (semi)simple Lie groups.*

Proof. Take the coordinate transformation

$$u = \frac{1}{\sqrt{2}}(x + y),$$
$$v = \frac{1}{\sqrt{2}}(x - y),$$

the symmetric tensor becomes

$$g' = du^2 - dv^2,$$

and the standard metric is

$$g = dx^2 + dy^2 = du^2 + dv^2.$$

The group of all $\mathbb{A} \otimes \mathbb{B}$-linear transformations of T^*W which preserve g' and the standard metric g is isomorphic to $G_n(m,l)^2$, the isotropy subgroup of W is obvious the group of all $\mathbb{A} \otimes \mathbb{B}$-linear transformations of W, that is $G_n(m,l)$. So the $G_n(m,l)^2$ orbit of W (with the Abelian part taken away) is the space in the table (C1) in the introduction.

Now We prove that the orbit is the connected component which contains W. Obviously the orbit is contained in the connected component, we only need to compare their dimensions. W is characterized by (I, I) under the coordinates $\{u, v\}$, a point

$$\begin{pmatrix} I & B \end{pmatrix},$$

is invariant under $\sigma_{g'}$ implies

$$(I, B)g' \left(I, \overline{B}\right)^t = 0,$$

where $B \in M_n(\mathbb{A} \otimes \mathbb{B})$. By simple calculation we have

$$B\overline{B}^t = I.$$

This implies that the dimension of the $G_n(m,l)^2$ orbit is the same as the connected component. □

Corollary 3.5.1 *When* $\mathbb{A},\mathbb{B}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$*, we have*

$$G_{\mathbb{AB}}(n)=G/K,$$

where

$$\begin{aligned} Lie\,(G) &= (\mathfrak{g}_n(m,l))^2, \textit{ and} \\ Lie\,(K) &= \mathfrak{g}_n(m,l). \end{aligned}$$

When $\mathbb{A}$ or/and $\mathbb{B}$ equals $\mathbb{O}$, the definition for $G_{\mathbb{AB}}(n)$ is naturally extended as below.

Theorem 3.5.1 *(Definition-Theorem): Suppose* $\mathbb{A}$ *and* $\mathbb{B}$ *are any normed division algebras.*

We define $G_{\mathbb{AB}}(n)$ *to be the symmetric space* G/K *with*

$$\begin{aligned} Lie\,(G) &= L_n(\mathbb{A},\mathbb{B})^2 = (Der(\mathbb{A})\oplus Der(\mathbb{B})\oplus A'_n(\mathbb{A}\otimes\mathbb{B}))^2 \\ Lie\,(K) &= L_n(\mathbb{A},\mathbb{B}), \end{aligned}$$

such that if $\mathbb{A},\mathbb{B}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$*, then* $G_{\mathbb{AB}}(n)$ *is the connected component containing* $(\mathbb{A}\otimes\mathbb{B})^n$ *of (with the Abelian part taken away)*

$$\bigcap_{I\in\Sigma^L\cup\Sigma^R\cup\{\sigma_{g'}\}} Gr_{\mathbb{R}}\left(\overline{k},T^*(\mathbb{A}\otimes\mathbb{B})^n\right)^{\sigma_I},$$

where $\overline{k}=\frac{dim_{\mathbb{R}}T^*(\mathbb{A}\otimes\mathbb{B})^n}{2}=n\cdot 2^{m+l}$.

If $\mathbb{A}$ *or* $\mathbb{B}$ *equals* $\mathbb{O}$*, then we assume that* n *equals* 3.

Proof. This follows immediately from proposition 3.5.1 and its corollary 3.5.1. □

Theorem 3.5.2 *For any normed division algebras* $\mathbb{A},\mathbb{B}\in\{\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}\}$*,* $G_{\mathbb{AB}}(n)$ *are given by the spaces in the table (C1) in the introduction.*

Chapter 4

Noncompact Riemannian Symmetric Spaces

4.1 Dual Riemannian Symmetric Spaces and Borel Embedding

Riemannian symmetric spaces of semisimple type come in pairs. Given any simply connected Riemannian symmetric space $X = G/K$, we have a decomposition of its orthogonal symmetric Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then $\mathfrak{g}^* = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{p} \subseteq \mathfrak{g}_{\mathbb{C}}$ is also an orthogonal symmetric Lie algebra and its corresponding symmetric space $\check{X}$ is called the *dual* of X. Furthermore X is compact if and only if $\check{X}$ is noncompact and vice versa. Another amazing property is the existence of a natural embedding $\check{X} \subset X$ as an open submanifold if X is a hermitian symmetric space, which is called the Borel embedding. For example, when $X = \mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(k)\mathrm{U}(n-k))$, then we have $\check{X} = \mathrm{SU}(k, n-k)/\mathrm{S}(\mathrm{U}(k)\mathrm{U}(n-k))$.

In this chapter we want to generalize the Borel embedding to other cases of pairs of Riemannian symmetric spaces.

4.2 An Embedding of Noncompact Symmetric Space into Its Compact Dual

In fact, for any symmetric space X, (not necessary hermitian) there exists a natural embedding $\check{X} \subset X$ as an open submanifold, at least for the classical cases. For example, when X is the Grassmannian $Gr_{\mathbb{R}}(k, W) = \mathrm{O}(N)/\mathrm{O}(k)\,\mathrm{O}(N-k)$, then we have $\check{X} = \mathrm{O}(k, N-k)/\mathrm{O}(k)\,\mathrm{O}(N-k)$. In fact $\check{X}$ is the Grassmannian of *spacelike* linear subspaces in W. To explain it, we need the following definitions.

Definition 4.2.1 *We assume W is a real vector space with a positive definite inner product g_W. Given a fixed linear subspace P_0 in W, which corresponds to an element P_0 in $Gr_{\mathbb{R}}(k, W)$, we have an orthogonal decomposition $W = P_0 \oplus P_0^{\perp}$ with respect to g_W. This determines an indefinite inner product $\check{g}_W$ on W given by*

$$\check{g}_W = g_W|_{P_0} - g_W|_{P_0^{\perp}}.$$

(i) An element P in $Gr_{\mathbb{R}}(k, W)$ is called spacelike if the restriction of $\check{g}_W$ to P is positive definite, i.e. $\check{g}_W|_P > 0$.

(ii) We denote $Gr^+_{\mathbb{R}}(k, W)$ the space of all spacelike elements in $Gr_{\mathbb{R}}(k, W)$.

(iii) For any subspace X of $Gr_{\mathbb{R}}(k, W)$ containing P_0, we define $X^+ = X \cap Gr^+_{\mathbb{R}}(k, W)$.

It is not difficult to see that the noncompact dual to the Grassmannian is precisely given by

$$Gr^+_{\mathbb{R}}(k, W) \cong \mathrm{O}(k, N-k)/\mathrm{O}(k)\,\mathrm{O}(N-k).$$

In this section, we are going to apply our descriptions of compact symmetric spaces as certain types of Grassmannians, we will show that under the above embedding the noncompact dual to any compact symmetric space X always consists of spacelike elements in X.

The following theorem generalizes the Borel embedding to any pair of Riemannian symmetric spaces of semisimple type in the classical cases.

Theorem 4.2.2 *Let* $W = (\mathbb{A} \otimes \mathbb{B})^n$*, where* $\mathbb{A}, \mathbb{B} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$*.*

(1) For any Grassmannian $Gr_{\mathbb{AB}}(k,n) \subset Gr_{\mathbb{R}}(k \cdot 2^{m+l}, W)$*, we consider* $P_0 = (\mathbb{A} \otimes \mathbb{B})^k \subset W$*, then* $Gr^+_{\mathbb{AB}}(k,n)$ *is the noncompact dual to* $Gr_{\mathbb{AB}}(k,n)$ *and the natural inclusion* $Gr^+_{\mathbb{AB}}(k,n) \subset Gr_{\mathbb{AB}}(k,n)$ *is an open embedding.*

(2) For any Lagrangian Grassmannian $LGr_{\mathbb{AB}}(n) \subset Gr_{\mathbb{R}}(n \cdot 2^{m+l-1}, W)$*, we consider* $P_0 = (\frac{\mathbb{A}}{2} \otimes \mathbb{B})^n \subset W$*, then* $LGr^+_{\mathbb{AB}}(n)$ *is the noncompact dual to* $LGr_{\mathbb{AB}}(n)$ *and the natural inclusion* $LGr^+_{\mathbb{AB}}(n) \subset LGr_{\mathbb{AB}}(n)$ *is an open embedding.*

(3) For any Double Lagrangian Grassmannian $LLGr_{\mathbb{AB}}(n) \subset Gr_{\mathbb{R}}(n \cdot 2^{m+l-1}, W)$*, we consider* $P_0 = (\frac{\mathbb{A}}{2} \otimes \frac{\mathbb{B}}{2})^n \oplus (j_1 \frac{\mathbb{A}}{2} \otimes j_2 \frac{\mathbb{B}}{2})^n$*, then* $LLGr^+_{\mathbb{AB}}(n)$ *is the noncompact dual to* $LLGr_{\mathbb{AB}}(n)$ *and the natural inclusion* $LLGr^+_{\mathbb{AB}}(n) \subset LLGr_{\mathbb{AB}}(n)$ *is an open embedding.*

(4) For any compact Lie group $G = G_{\mathbb{AB}}(n) \subset Gr_{\mathbb{R}}(n \cdot 2^{m+l}, T^*W)$*, we consider* $P_0 = W$*, then* $G^+_{\mathbb{AB}}(n)$ *is the noncompact dual to* $G_{\mathbb{AB}}(n)$*, and the natural inclusion* $G^+_{\mathbb{AB}}(n) \subset G_{\mathbb{AB}}(n)$ *is an open embedding.*

Proof. Let X be a compact Riemannian symmetric space, recall that X is the fix point set of some involutive isometries of $Gr_{\mathbb{R}}(k, W)$. We want to prove

1. These involutions can be restricted to $Gr^+_{\mathbb{R}}(k, W)$.
2. These involutions are isometries on $Gr^+_{\mathbb{R}}(k, W)$ with respect to the invariant metric on $Gr^+_{\mathbb{R}}(k, W)$.

Then 1 and 2 implies that M^+ is a noncompact Riemannian symmetric space sitting in M as an open subset by lemma 3.1.1.

Recall the involutive isometries which we used in chapter 3 are σ_J σ_ω and $\sigma_{g'}$, we will prove these two facts for σ_J σ_ω and $\sigma_{g'}$ separately. To simplify the notations in the proof we assume that the dimension of W is $2n$ in the cases of

σ_J and σ_ω and the dimension of T^*W is $2n$ in the case of $\sigma_{g'}$, the dimension of P_0 is $2k$ in the case of σ_J and n in the cases of σ_ω and $\sigma_{g'}$.

1. For the involution σ_J induced by a complex structure J, take an orthogonal bases $e_1, e_2, ..., e_{2n}$ of W, such that for each i, $Je_{2i-1} = e_{2i}, Je_{2i} = -e_{2i-1}$. Then as a matrix $J \in \mathrm{O}(2k)\mathrm{O}(2n-2k)$. For any $A \in \mathrm{O}(2k, 2n-2k)$, define

$$\begin{aligned} \check{\sigma}_J : Gr^+_{\mathbb{R}}(2k, W) &\rightarrow Gr^+_{\mathbb{R}}(2k, W), \\ \check{\sigma}_J(AK) &= AJK, \end{aligned}$$

where K is $\mathrm{O}(2k)\mathrm{O}(2n-2k)$. It is easy to check $\check{\sigma}_J$ is well-defined. The following natural embedings

$$\mathrm{O}(2n) \hookrightarrow \mathrm{GL}(2n, \mathbb{R})$$

and

$$\mathrm{O}(2k, 2n-2k) \hookrightarrow \mathrm{GL}(2n, \mathbb{R})$$

induce the natural embeddings

$$\mathrm{O}(2k, 2n-2k)/\mathrm{O}(2k)\mathrm{O}(2n-2k) \hookrightarrow \mathrm{GL}(2n, \mathbb{R})/P$$

and

$$\mathrm{O}(2n)/\mathrm{O}(2k)\mathrm{O}(2n-2k) \hookrightarrow \mathrm{GL}(2n, \mathbb{R})/P,$$

where

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathrm{GL}(2n, \mathbb{R}) \mid a, b, c \in M_n(\mathbb{R}) \right\}.$$

For any matrix A in $\mathrm{GL}(2n, \mathbb{R})$, one can use the Gram-Schmidt process to find a matrix B in P, such that AB is an orthogonal matrix. This implies that the above last embedding is surjective, i.e. $\mathrm{GL}(2n, \mathbb{R})/P$ can be identified as $\mathrm{O}(2n)/\mathrm{O}(2k)\,\mathrm{O}(2n-2k)$. Let $B' = J^{-1}BJ \in P$,

$$\begin{aligned}
\sigma_J(ABK) &= ABJK \quad (\text{in } \mathrm{O}(2n)/\mathrm{O}(2k)\,\mathrm{O}(2n-2k)) \\
&= AJB'K \quad (\text{in } \mathrm{O}(2n)/\mathrm{O}(2k)\,\mathrm{O}(2n-2k)) \\
&= AJP \qquad (\text{in } \mathrm{GL}(2n,\mathbb{R})/P) \\
&= AJK \qquad (\text{in } \mathrm{O}(2k,2n-2k)/\mathrm{O}(2k)\,\mathrm{O}(2n-2k)) \\
&= \check{\sigma}_J(AK) \quad (\text{in } \mathrm{O}(2k,2n-2k)/\mathrm{O}(2k)\,\mathrm{O}(2n-2k)),
\end{aligned}$$

this implies $\check{\sigma}_J = \sigma_J \mid_{Gr^+_{\mathbb{R}}(k,W)}$. $\check{\sigma}_J$ is isometry on $Gr^+_{\mathbb{R}}(k,W)$ because J is a matrix in $\mathrm{O}(2k)\mathrm{O}(2n-2k)$.

2. For the involution σ_ω induced by a symplectic structure ω, take an orthogonal bases $e_1, e_2, ..., e_{2n}$ of W, such that ω is of the form $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ as a non-degenerate tow form. Denote by D the matrix $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, where I is the identity matrix of rank n. For any $A \in \mathrm{O}(n,n)$, define

$$\begin{aligned}
\check{\sigma}_\omega : Gr^+_{\mathbb{R}}(n,W) &\rightarrow Gr^+_{\mathbb{R}}(n,W), \\
\check{\sigma}_\omega(AK) &= DAD^{-1}K,
\end{aligned}$$

where K is $\mathrm{O}(n)\mathrm{O}(n)$. $\check{\sigma}_\omega$ is well-defined.

A in $\mathrm{O}(n,n)$ means

$$A \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} A^t = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

The Gram-Schmidt process asserts that there is a matrix B in P such that AB is an orthogonal matrix, that is

$$(AB)^t(AB) = I.$$

B is a matrix in P, hence the matrix $A^tAB = (B^t)^{-1} = (B^{-1})^t$ is of the form

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}.$$

This implies that

$$\begin{aligned} DA^{-1}D^{-1}\omega AB\omega^{-1} &= DA^{-1}\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}AB\omega^{-1} \\ &= D\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}A^tAB\omega^{-1} \\ &= \omega A^tAB\omega^{-1} \\ &= \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix} \end{aligned}$$

is in P.

$$\begin{aligned} \sigma_\omega(ABK) &= \omega AB\omega^{-1}K \quad (\text{in } \mathrm{O}(2n)/\mathrm{O}(n)\,\mathrm{O}(n)) \\ &= (DAD^{-1})(DA^{-1}D^{-1}\omega AB\omega^{-1})K \quad (\text{in } \mathrm{O}(2n)/\mathrm{O}(n)\,\mathrm{O}(n)) \\ &= DAD^{-1}P \qquad (\text{in } \mathrm{GL}(2n,\mathbb{R})/P) \\ &= DAD^{-1}K \qquad (\text{in } \mathrm{O}(n,n)/\mathrm{O}(n)\,\mathrm{O}(n)) \\ &= \check{\sigma}_\omega(AK) \quad (\text{in } \mathrm{O}(n,n)/\mathrm{O}(n)\,\mathrm{O}(n)) \end{aligned}$$

implies that $\check{\sigma}_\omega = \sigma_\omega \mid_{Gr^+_{\mathbb{R}}(k,W)}$. Let $X = \begin{pmatrix} 0 & b \\ b^t & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & c \\ c^t & 0 \end{pmatrix}$ be vectors in the tangent space of $Gr^+_{\mathbb{R}}(k,W)$ at P_0, $\check{g}(X,Y) = \frac{1}{2}Tr(XY)$ the invariant metric on $Gr^+_{\mathbb{R}}(n,W)$. The isometry of $\check{\sigma}_\omega$ follows from

$$\begin{aligned} \check{g}(X,Y) &= \frac{1}{2}Tr(XY) \\ &= \frac{1}{2}Tr\left(\begin{pmatrix} 0 & b \\ b^t & 0 \end{pmatrix}\begin{pmatrix} 0 & c \\ c^t & 0 \end{pmatrix}\right) \\ &= \frac{1}{2}Tr(bc^t + b^tc), \end{aligned}$$

and

$$\begin{aligned}
\check{g}((\check{\sigma}_\omega)_* X, (\check{\sigma}_\omega)_* Y) &= \frac{1}{2} Tr((\check{\sigma}_\omega)_*(X) \cdot (\check{\sigma}_\omega)_*(Y)) \\
&= \frac{1}{2} Tr\left(\begin{pmatrix} 0 & b^t \\ b & 0 \end{pmatrix}\begin{pmatrix} 0 & c^t \\ c & 0 \end{pmatrix}\right) \\
&= \frac{1}{2} Tr(b^t c + b c^t).
\end{aligned}$$

3. For the involution $\sigma_{g'}$ induced by a symmetric 2-tensor g', take an orthogonal bases $e_1, e_2, ..., e_{2n}$ of T^*W, such that g' is of the form $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. Denote $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ by Ω, where I is the identity matrix of rank n. For any $A \in \mathrm{O}(n,n)$, define

$$\begin{aligned}
\check{\sigma}_{g'} : Gr^+_{\mathbb{R}}(n, T^*W) &\rightarrow Gr^+_{\mathbb{R}}(n, T^*W), \\
\check{\sigma}_{g'}(AK) &= \Omega A \Omega^{-1} K,
\end{aligned}$$

where K is $\mathrm{O}(n)\mathrm{O}(n)$. $\check{\sigma}_{g'}$ is well-defined. Use the notations as above,

$$\begin{aligned}
\sigma_{g'}(ABK) &= g' A B g'^{-1} K \ \ (\text{in } \mathrm{O}(2n)/\mathrm{O}(n)\mathrm{O}(n)) \\
&= (\Omega A \Omega^{-1})(\Omega A^{-1} \Omega^{-1} g' A B g'^{-1}) K \ (\text{in } \mathrm{O}(2n)/\mathrm{O}(n)\mathrm{O}(n)) \\
&= (\Omega A \Omega^{-1})(g' A^t A B g') K \ (\text{in } \mathrm{O}(2n)/\mathrm{O}(n)\mathrm{O}(n)) \\
&= (\Omega A \Omega^{-1}) \begin{pmatrix} c & b \\ 0 & a \end{pmatrix} K \ (\text{in } \mathrm{O}(2n)/\mathrm{O}(n)\mathrm{O}(n)) \\
&= \Omega A \Omega^{-1} P \ (\text{in } \mathrm{GL}(2n, \mathbb{R})/P) \\
&= \Omega A \Omega^{-1} K \ (\text{in } \mathrm{O}(n,n)/\mathrm{O}(n)\mathrm{O}(n)) \\
&= \check{\sigma}_{g'}(AK) \ (\text{in } \mathrm{O}(n,n)/\mathrm{O}(n)\mathrm{O}(n))
\end{aligned}$$

implies that $\check{\sigma}_{g'} = \sigma_{g'} \mid_{Gr^+_{\mathbb{R}}(k,W)}$. Let X, Y be the above tangent vectors of $Gr^+_{\mathbb{R}}(k, W)$ at P_0, $\check{g}(X, Y)$ the invariant metric as before. The isometry of

$\check{\sigma}_{g'}$ follows from

$$\begin{aligned}
\check{g}((\check{\sigma}_{g'})_* X, (\check{\sigma}_{g'})_* Y) &= \frac{1}{2} Tr((\check{\sigma}_{g'})_*(X) \cdot (\check{\sigma}_{g'})_*(Y)) \\
&= \frac{1}{2} Tr\left(\begin{pmatrix} 0 & -b^t \\ -b & 0 \end{pmatrix}\begin{pmatrix} 0 & -c^t \\ -c & 0 \end{pmatrix}\right) \\
&= \frac{1}{2} Tr(b^t c + b c^t) \\
&= \check{g}(X, Y).
\end{aligned}$$

□

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